

# THE CUT-OFF PHENOMENON FOR BROWNIAN MOTIONS ON SYMMETRIC SPACES OF COMPACT TYPE

PIERRE-LOÏC MÉLIOT

**ABSTRACT.** In this paper, we prove the cut-off phenomenon in total variation distance for the Brownian motions traced on the classical symmetric spaces of compact type, that is to say:

- (1) the classical simple compact Lie groups: special orthogonal groups  $SO(n)$ , special unitary groups  $SU(n)$  and compact symplectic groups  $USp(n)$ ;
- (2) the real, complex and quaternionic Grassmannian varieties (including the real spheres and complex or quaternionic projective spaces when  $q = 1$ ):  $SO(p+q)/(SO(p) \times SO(q))$ ,  $SU(p+q)/(U(p) \times U(q))$  and  $USp(p+q)/(USp(p) \times USp(q))$ ;
- (3) the spaces of structures:  $SU(n)/SO(n)$ ,  $SO(2n)/U(n)$ ,  $SU(2n)/USp(n)$ , and  $USp(n)/U(n)$ .

In each case, we give explicit lower bounds for  $d_{TV}(\mu_t, \text{Haar})$  if  $t < t_{\text{cut-off}} = \alpha \log n$ , and explicit upper bounds if  $t > t_{\text{cut-off}}$ . This gives in particular an answer to some questions raised in [SC04, CSC08]. Our proofs are mainly inspired by those of the papers [CSST08, Ros94, Por96a, Por96b], in which the same kind of results is shown for products of random transpositions in  $\mathfrak{S}_n$ , and for products of random rotations in  $SO(n)$ .

---

## CONTENTS

1. Introduction	3
1.1. The cut-off phenomenon for random permutations	3
1.2. Classical compact Lie groups and symmetric spaces	4
1.3. Laplace-Beltrami operators and Brownian motions on symmetric spaces	6
1.4. Chen-Saloff-Coste results on $\mathcal{L}^p$ -cut-offs of Markov processes	11
1.5. Statement of the main results and discriminating events	12
Organization of the paper	15
Acknowledgements	15
2. Fourier expansion of the densities	16
2.1. Peter-Weyl's theorem and Cartan's refinement	16
2.2. Highest weight theorem and Weyl's character formula	18
2.3. Densities of a Brownian motion with values in a compact symmetric space	20
3. Upper bounds after the cut-off time	29
3.1. Guessing the order of decay of the dominating series	32
3.2. Growth of the dimensions versus decay of the Laplace-Beltrami eigenvalues	35
3.2.1. Symplectic groups and their quotients	35
3.2.2. Odd special orthogonal groups and their quotients	41
3.2.3. Even special orthogonal groups and their quotients	44
3.2.4. Special unitary groups and their quotients	45
4. Lower bounds before the cut-off time	49
4.1. Expansion of the square of the discriminating zonal spherical functions	52
4.1.1. Values of the zonal functions and abstract expansions of their squares	54
4.1.2. Quotients of orthogonal groups	59
4.1.3. Quotients of unitary groups	65

---

*Date:* October 2, 2012.

4.1.4. Quotients of symplectic groups	68
4.2. Proof of the lower bound on the total variation distance	72
References	77

---

## 1. INTRODUCTION

**1.1. The cut-off phenomenon for random permutations.** This paper is concerned with the analogue for Brownian motions on compact Lie groups and symmetric spaces of the famous *cut-off phenomenon* observed in random shuffles of cards (*cf.* [AD86, BD92]). Let us recall this very beautiful result in the case of “natural” shuffles of cards, also known as *riffle shuffles*. Consider a deck of  $n$  ordered cards  $1, 2, \dots, n$ , originally in this order. At each time  $k \geq 1$ , one performs the following procedure:

- (1) One cuts the deck in two parts of sizes  $m$  and  $n - m$ , the integer  $m$  being chosen randomly according to a binomial law of parameter  $\frac{1}{2}$ :

$$\mathbb{P}[m = M] = \frac{1}{2^n} \binom{n}{M}.$$

So for instance, if  $n = 10$  and the deck was initially 123456789X, then one obtains the two blocks  $A = 123456$  and  $B = 789X$  with probability  $\frac{1}{2^{10}} \binom{10}{6} = \frac{105}{512} \simeq 0.21$ .

- (2) The first card of the new deck comes from  $A$  with probability  $(\text{card } A)/n$ , and from  $B$  with probability  $(\text{card } B)/n$ . Then, if  $A'$  and  $B'$  are the remaining blocks after removal of the first card, the second card of the new deck will come from  $A'$  with probability  $(\text{card } A')/(n - 1)$ , and from  $B'$  with probability  $(\text{card } B')/(n - 1)$ ; and similarly for the other cards. So for instance, by shuffling  $A = 123456$  and  $B = 789X$ , one can obtain with probability  $1/\binom{10}{6} \simeq 0.0048$  the deck 17283459X6.

An equivalent description of this shuffle of cards is as follows: one chooses a part  $P$  of  $\llbracket 1, n \rrbracket$  with uniform probability  $\frac{1}{2^n}$ , and if  $M = \text{card } P$ , then one inserts the  $M$  first cards of the deck in  $P$ , and the  $n - M$  last cards of the deck in  $\llbracket 1, n \rrbracket \setminus P$ . Denote  $\mathfrak{S}_n$  the symmetric group of order  $n$ , and  $\sigma^{(k)}$  the random permutation in  $\mathfrak{S}_n$  obtained after  $k$  independent shuffles. One can guess that as  $k$  goes to infinity, the law  $\mathbb{P}^{(k)}$  of  $\sigma^{(k)}$  converges to the uniform law on  $\mathfrak{S}_n$ , that is to say that for any fixed permutation  $\sigma$  one has

$$\mathbb{P}[\sigma^{(k)} = \sigma] = \mathbb{P}^{(k)}[\sigma] \xrightarrow{k \rightarrow \infty} \mathbb{U}[\sigma] = \frac{1}{n!}.$$

There is a natural distance on the set  $\mathcal{P}(\mathfrak{S}_n)$  of probability measures on  $\mathfrak{S}_n$  that allows to measure this convergence: the so-called *total variation distance*  $d_{\text{TV}}$ . Consider more generally a measurable space  $X$  with  $\sigma$ -field  $\mathcal{B}(X)$ . The total variation distance is the metric on the set of probability measures  $\mathcal{P}(X)$  defined by

$$d_{\text{TV}}(\mu, \nu) = \sup \{ |\mu(A) - \nu(A)|, A \in \mathcal{B}(X) \} \in [0, 1].$$

Notice that the convergence in total variation distance is a much stronger notion than the weak convergence of probability measures: for instance, if  $(x_n)_{n \in \mathbb{N}}$  is a sequence of real numbers converging to some  $x \in \mathbb{R}$ , then the Dirac measures  $\delta_{x_n}$  converge weakly to  $\delta_x$ , whereas  $d_{\text{TV}}(\delta_{x_n}, \delta_x)$  is equal to 1 for any  $n \in \mathbb{N}$ . On the other hand, if  $\mu$  and  $\nu$  are absolutely continuous with respect to a third measure  $dx$  on  $X$ , then their total variation distance can be written as a  $\mathcal{L}^1$ -norm:

$$d_{\text{TV}}(\mu, \nu) = \frac{1}{2} \int_X \left| \frac{d\mu}{dx}(x) - \frac{d\nu}{dx}(x) \right| dx.$$

That said, let us return to the convergence of shuffles. It turns out that with respect to total variation distance, the convergence occurs at a specific time  $k_{\text{cut-off}}$ , that is to say that  $d_{\text{TV}}(\mathbb{P}^{(k)}, \mathbb{U})$  stays close to 1 for  $k < k_{\text{cut-off}}$ , and that  $d_{\text{TV}}(\mathbb{P}^{(k)}, \mathbb{U})$  is then extremely close to 0 for  $k > k_{\text{cut-off}}$ . More precisely, in [BD92] (see also [CSST08, Chapter 10]), it is shown that:

**Theorem 1** (Bayer-Diaconis). *Suppose  $k = \frac{3}{2 \log 2} \log n + \theta$ . Then,*

$$d_{\text{TV}}(\mathbb{P}^{(k)}, \mathbb{U}) = 1 - 2 \phi \left( \frac{-2^{-\theta}}{4\sqrt{3}} \right) + O(n^{-1/4}), \quad \text{with } \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{s^2}{2}} ds.$$

So for  $\theta$  negative, the total variation distance is extremely close to 1, whereas it is extremely close to 0 for  $\theta$  positive.

When  $n = 52$ , the cut-off time is  $\frac{3 \log 52}{2 \log 2} \sim 8.55$ , so “8 shuffles<sup>1</sup> are sufficient to obtain an almost uniform permutation of the cards of a standard deck”. The cut-off phenomenon has been proved for other shuffling algorithms (*e.g.* random transpositions of cards), and more generally for large classes of finite Markov chains, see for instance [DSC96, Dia96]. It has also been investigated by Chen and Saloff-Coste for Markov processes on continuous spaces, *e.g.* spheres and Lie groups; see in particular [SC94, SC04, CSC08] and the discussion of §1.4. However, in this case, cut-offs are easier to prove for the  $\mathcal{L}^{p>1}$ -norm of  $p_t(x) - 1$ , where  $p_t(x)$  is the density of the process at time  $t$  and point  $x$  with respect to the equilibrium measure. The case of the  $\mathcal{L}^1$ -norm, which is (up to a factor 2) the total variation distance, is somewhat different. In particular, a proof of the cut-off phenomenon for the total variation distance between the Haar measure and the marginal law  $\mu_t$  of the Brownian motion on a classical compact Lie group (say, a unitary group) was apparently not known — see the remark just after [CSC08, Theorem 1.2]. The purpose of this paper is precisely to give a proof of this  $\mathcal{L}^1$ -cut-off for all classical compact Lie groups (special orthogonal groups, special unitary groups, and compact symplectic groups), and more generally for all classical symmetric spaces of compact type.

In the two next paragraphs, we describe the spaces in which we will be interested (§1.2), and we precise what is meant by “Brownian motion” on a space of this type (*cf.* §1.3). This will then enable us to explain the results of Chen and Saloff-Coste in §1.4, and finally to state in §1.5 which improvements we were able to prove.

**1.2. Classical compact Lie groups and symmetric spaces.** To begin with, let us fix some notations regarding the three classical families of simple compact Lie groups, and their quotients corresponding to irreducible simply connected compact symmetric spaces. We follow here most of the conventions of [Hel78, Hel84]. For every  $n \geq 1$ , we denote  $U(n) = U(n, \mathbb{C})$  the *unitary group* of order  $n$ ; this is the set of complex matrices of size  $n \times n$  such that

$$UU^\dagger = U^\dagger U = I_n.$$

Here,  $U^\dagger = \overline{U}^t$  is the complex conjugate of the transpose of  $U$ . Similarly, the *orthogonal group* of order  $n$  is the set  $O(n) = O(n, \mathbb{R})$  of real matrices of size  $n \times n$  such that

$$OO^t = O^t O = I_n;$$

and the *compact symplectic group* of order  $n$  is the set  $USp(n) = USp(n, \mathbb{H})$  of quaternionic matrices of size  $n \times n$  such that

$$SS^* = S^* S = I_n.$$

Here,  $S^* = \overline{S}^t$  denotes the quaternionic conjugate of the transpose of  $S$ , the conjugate of a quaternion  $w + ix + jy + kz$  being  $w - ix - jy - kz$ . The orthogonal groups are not connected, so we shall rather work with the *special orthogonal groups*

$$SO(n) = SO(n, \mathbb{R}) = \{O \in O(n, \mathbb{R}) \mid \det O = 1\}.$$

On the other hand, the unitary groups are not simple Lie groups (their center is one-dimensional), so it is convenient to introduce the *special unitary groups*

$$SU(n) = SU(n, \mathbb{C}) = \{U \in U(n, \mathbb{C}) \mid \det U = 1\}.$$

Then, for every  $n \geq 1$ ,  $SU(n, \mathbb{C})$ ,  $SO(n, \mathbb{R})$  and  $USp(n, \mathbb{H})$  are connected simple compact real Lie groups, of respective dimensions

$$\dim_{\mathbb{R}} SU(n, \mathbb{C}) = n^2 - 1 \quad ; \quad \dim_{\mathbb{R}} SO(n, \mathbb{R}) = \frac{n(n-1)}{2} \quad ; \quad \dim_{\mathbb{R}} USp(n, \mathbb{H}) = 2n^2 + n$$

---

<sup>1</sup>Actually the common say is that *seven* shuffles are sufficient; indeed the  $O(n^{-1/4})$  is quite big for  $n = 52$ , and one can compute  $d_{TV}(\mathbb{P}^{(6)}, \mathbb{U}) \sim 0.61$ ,  $d_{TV}(\mathbb{P}^{(7)}, \mathbb{U}) \sim 0.33$  and  $d_{TV}(\mathbb{P}^{(8)}, \mathbb{U}) \sim 0.17$ . We shall explain later why the total variation distance is decreasing with  $k$ ; see Lemma 5.

and with respective Lie algebras

$$\begin{aligned}\mathfrak{su}(n) &= \mathfrak{su}(n, \mathbb{C}) = \{M \in \mathbf{M}(n, \mathbb{C}) \mid M^\dagger + M = 0 \text{ and } \operatorname{tr} M = 0\}; \\ \mathfrak{so}(n) &= \mathfrak{so}(n, \mathbb{R}) = \{M \in \mathbf{M}(n, \mathbb{R}) \mid M^t + M = 0\}; \\ \mathfrak{usp}(n) &= \mathfrak{sp}(n, \mathbb{H}) = \{M \in \mathbf{M}(n, \mathbb{H}) \mid M^\star + M = 0\}.\end{aligned}$$

The special unitary groups and compact symplectic groups are simply connected; on the other hand, for  $n \geq 3$ , the fundamental group of  $\mathrm{SO}(n, \mathbb{R})$  is  $\mathbb{Z}/2\mathbb{Z}$ , and its universal cover is the *spin group*  $\mathrm{Spin}(n)$ .

Many computations on these simple compact Lie groups can be performed by using their *representation theory*, which is covered by the highest weight theorem; see Theorem 12 in §2.2. We shall recall all this briefly in Section 2, and give in each case the list of all irreducible representations, and the corresponding dimensions and characters. It is well known that every simply connected compact simple Lie group is:

- either one group in the infinite families  $\mathrm{SU}(n)$ ,  $\mathrm{Spin}(n)$ ,  $\mathrm{USp}(n)$ ;
- or, an exceptional simple compact Lie group of type  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  or  $G_2$ .

We shall refer to the first case as the *classical simple compact Lie groups*, and as mentioned before, our goal is to study Brownian motions on these groups.

We shall more generally be interested in compact symmetric spaces; let us recall briefly what they are. A locally symmetric space is a (connected) Riemannian manifold  $X$  such that every point  $x \in X$  has a symmetric neighborhood  $N_x$ , on which is defined an isometry  $s : N_x \rightarrow N_x$  with

$$s(\gamma(t)) = \gamma(-t)$$

for every geodesic  $\gamma$  such that  $\gamma(0) = x$ . This is equivalent to the vanishing of the torsion and the invariance of the curvature tensor by parallel transport (see [Hel78, Chapter 4]). A *symmetric space* is a locally symmetric space whose local geodesic symmetries can be extended to global isometries of  $X$ . In that case, the group of isometries of  $X$  endowed with the compact-open topology is in fact a real Lie group, and if  $G$  is the connected component of  $\mathrm{id}_X$  in  $\mathrm{Isom}(X)$ , then:

**Theorem 2** (Cartan-Ambrose-Hicks). *The space  $X$  is isomorphic (as a Riemannian manifold) to  $G/K$ , where  $K$  is the stabilizer of a point  $x \in X$  and a compact subgroup of  $G$ ; and  $(G, K)$  is a symmetric pair, which means that  $K$  is included in the group of fixed points  $G^\theta$  of an involutive automorphism of  $G$ , and contains the connected component  $(G^\theta)^0$  of the identity in this group. Moreover,  $X$  is compact if and only if  $G$  is compact.*

This result reduces the classification of symmetric spaces to the classification of real Lie groups and their involutive automorphisms. More precisely, consider a simply connected symmetric space  $X$  — the simple connectedness can be forced by taking the universal cover of the manifold. It is always a direct product of  $X_1 \times X_2 \times \cdots \times X_k$  of *irreducible* simply connected symmetric spaces. So, consider an irreducible simply connected symmetric space, of compact type. Two cases arise:

- (1) The isometry group  $G = K \times K$  is the product of a compact simple Lie group with itself, and  $K$  is embedded into  $G$  via the diagonal map  $k \mapsto (k, k)$ . The symmetric space  $X$  is then the group  $K$  itself, the quotient map from  $G$  to  $X \simeq K$  being

$$\begin{aligned}G &\rightarrow K \\ g = (k_1, k_2) &\mapsto k_1 k_2^{-1}.\end{aligned}$$

In particular, the isometries of  $K$  are the multiplication on the left and the right by elements of  $K \times K$ , and this action restricted to  $K \subset G$  is the action by conjugacy.

- (2) The isometry group  $G$  is a compact simple Lie group, and  $K$  is a closed subgroup of it. In this case, there exists in fact a non-compact simple Lie group  $L$  with maximal compact subgroup  $K$ , such that  $G$  is a compact subgroup of the complexified Lie group  $L^\mathbb{C}$ , and maximal among those

containing  $K$ . The involutive automorphism  $\theta$  extends to  $L^\mathbb{C}$ , with  $K = G^\theta = L^\theta$  and the two orthogonal symmetric Lie algebras  $(\mathfrak{g}, d_e\theta)$  and  $(\mathfrak{l}, d_e\theta)$  dual of each other.

The classification of irreducible simply connected compact symmetric spaces is therefore the following: in addition to the compact simple Lie groups themselves, there are the seven following infinite families

$$\begin{aligned} \text{Gr}(p+q, q, \mathbb{R}) &= \text{SO}(p+q)/(\text{SO}(p) \times \text{SO}(q)) \text{ with } p, q \geq 1 \text{ (real Grassmannians);} \\ \text{Gr}(p+q, q, \mathbb{C}) &= \text{SU}(p+q)/(\text{S}(\text{U}(p) \times \text{U}(q))) \text{ with } p, q \geq 1 \text{ (complex Grassmannians);} \\ \text{Gr}(p+q, q, \mathbb{H}) &= \text{USp}(p+q)/(\text{USp}(p) \times \text{USp}(q)) \text{ with } p, q \geq 1 \text{ (quaternionic Grassmannians);} \\ \text{SU}(n)/\text{SO}(n) &\text{ with } n \geq 2 \text{ (real structures on } \mathbb{C}^n); \\ \text{USp}(n)/\text{U}(n) &\text{ with } n \geq 1 \text{ (complex structures on } \mathbb{H}^n); \\ \text{SO}(2n)/\text{U}(n) &\text{ with } n \geq 2 \text{ (complex structures on } \mathbb{R}^{2n}); \\ \text{SU}(2n)/\text{USp}(n) &\text{ with } n \geq 2 \text{ (quaternionic structures on } \mathbb{C}^{2n}); \end{aligned}$$

and quotients involving exceptional Lie groups, *e.g.*  $\mathbb{P}^2(\mathbb{O}) = \text{F}_4/\text{Spin}(9)$ ; see [Hel78, Chapter 10]. For the two last families, one sees  $\text{U}(n)$  as a subgroup of  $\text{SO}(2n)$  by replacing each complex number  $x + iy$  by the  $2 \times 2$  real matrix

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}; \quad (1.1)$$

and one sees  $\text{USp}(n)$  as a subgroup of  $\text{SU}(2n)$  by replacing each quaternion number  $w + ix + jy + kz$  by the  $2 \times 2$  complex matrix

$$\begin{pmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{pmatrix}; \quad (1.2)$$

actually  $\text{USp}(n, \mathbb{H})$  is then the intersection of  $\text{SU}(2n, \mathbb{C})$  and of the complex symplectic group  $\text{Sp}(2n, \mathbb{C})$ . We shall refer to the seven aforementioned families as *classical simple compact symmetric spaces* (of type non-group); again, we aim to study in detail the Brownian motions on these spaces.

**1.3. Laplace-Beltrami operators and Brownian motions on symmetric spaces.** In the following, we denote  $d\eta_K(k)$  or  $dk$  the *Haar measure* of a (simple) compact Lie group  $K$ , and  $d\eta_X(x)$  or  $dx$  the Haar measure of a compact symmetric space  $X = G/K$ , which is the image measure of  $d\eta_G$  by the projection map  $\pi : G \rightarrow G/K$ . In both cases, this is the unique probability measure on the space that is  $G$ -invariant. We refer to [Hel84, Chapter 1] for precisions on the integration theory over (compact) Lie groups and their homogeneous spaces.

There are several complementary ways to define a Brownian motion on a compact Lie group  $K$  or a on compact symmetric space  $G/K$ , see in particular [Lia04b]. Hence, one can view them:

- (1) as Markov processes with infinitesimal generator the Laplace-Beltrami differential operator of the underlying Riemannian manifold;
- (2) as conjugacy-invariant continuous Lévy processes on  $K$ , or as projections of such a process on  $G/K$ ;
- (3) at least in the group case, as solutions of stochastic differential equations driven by standard (multidimensional) Brownian motions on the Lie algebra.

The first and the third point of view will be specially useful for our computations. For the sake of completeness, let us recall briefly each point of view — the reader already acquainted with these notions can thus go directly to §1.4.

To begin with, let us precise the Riemannian structures chosen in each case. We recall that the adjoint representations of a Lie group  $G$  and of its Lie algebra  $\mathfrak{g}$  are defined by

$$\begin{aligned} \text{Ad} : G &\rightarrow \text{GL}(\mathfrak{g}) \\ g &\mapsto (\text{Ad } g : X \mapsto \{d_e C_g\}(X)); \end{aligned} \quad (1.3)$$

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \text{End}(\mathfrak{g}) \\ X &\mapsto (\text{ad } X = \{d_e \text{Ad}\}(X)). \end{aligned} \quad (1.4)$$

Then the Lie bracket is  $[X, Y] = \text{ad } X(Y)$ . Given a simple compact Lie group  $K$ , there is up to a scalar constant a unique scalar product on the Lie algebra  $\mathfrak{k} = T_e K$  that is  $\text{Ad}(K)$ -invariant. A way to construct it is to take the mean  $\int_K \langle \text{Ad } k(\cdot) | \text{Ad } k(\cdot) \rangle dk$  of an arbitrary scalar product. Alternatively, one can take the opposite of the *Killing form*  $B(X, Y) = \text{tr}(\text{ad } X \circ \text{ad } Y)$ , which is negative definite for a simple compact group. Then, one can transport this scalar product  $\langle \cdot | \cdot \rangle_{\mathfrak{k}}$  on any tangent space  $\langle \cdot | \cdot \rangle_{T_k K}$ :

$$\langle X | Y \rangle_{T_k K} = \langle \{d_k L_{k^{-1}}\}(X) | \{d_k L_{k^{-1}}\}(Y) \rangle_{\mathfrak{k}}, \quad (1.5)$$

where  $L_{k^{-1}}$  is the multiplication on the left by  $k^{-1}$ . Thus one obtains a Riemannian structure on  $K$  which is invariant by  $G = K \times K$ , and up to a scalar the unique Riemannian structure with this property.

When  $K = \text{SU}(n)$  or  $\text{SO}(n)$  or  $\text{USp}(n)$ , the Killing form on  $\mathfrak{k}$  is a scalar multiple of the bilinear form  $X \otimes Y \mapsto \Re(\text{tr}(XY))$  — the real part is only needed for the quaternionic case. Following the conventions of [L  v11], we shall always consider the following invariant scalar products on  $\mathfrak{k}$ :

$$\langle X | Y \rangle = -\frac{\beta n}{2} \Re(\text{tr}(XY)), \quad (1.6)$$

with  $\beta = 1$  for special orthogonal groups,  $\beta = 2$  for special unitary groups and unitary groups, and  $\beta = 4$  for compact symplectic groups. Notice that the bilinear form in (1.6) is only proportional to minus the Killing form, and not equal to it; for instance, the Killing form of  $\text{SO}(n, \mathbb{R})$  is

$$(n-2) \text{tr}(XY) = -\frac{2n-4}{n} \langle X | Y \rangle,$$

and not  $-\langle X | Y \rangle$ . However, the normalization of Formula (1.6) becomes asymptotically the same as one-half the Killing form, and on the other hand, it enables one to relate the Brownian motions on the compact Lie groups to the “standard” Brownian motions on their Lie algebras, and to the classical ensembles of random matrix theory (see the stochastic differential equations at the end of this paragraph).

Similarly, suppose that  $X = G/K$  is a simple compact symmetric space (of type non-group). We take an  $\text{Ad}(G)$ -invariant scalar product  $\langle \cdot | \cdot \rangle_{\mathfrak{g}}$  on  $\mathfrak{g}$  (unique up to a scalar), and we denote  $\mathfrak{x}$  the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ . Notice that  $\mathfrak{x}$  can be identified with the tangent space of  $X = G/K$  at  $eK$  by the map  $d\pi_e$ , where  $\pi(g) = gK$ . One transports as in Equation (1.5) the restriction to  $\mathfrak{x}$  of the scalar product  $\langle \cdot | \cdot \rangle_{\mathfrak{g}}$  to any tangent space  $T_x X$ : one thus gets a  $G$ -invariant Riemannian structure on  $X$ , which is in fact unique up to a scalar constant. It is called the Riemannian structure *induced* by the Riemannian structure of  $G$ .

From now on, each classical simple compact symmetric space  $X = G/K$  will be endowed with the Riemannian structure induced by the Riemannian structure of  $G = \text{SU}(n)$  or  $\text{SO}(n)$  or  $\text{USp}(n)$  that was given by Equation (1.6). One may argue that this is not necessarily the best choice of normalization for these quotients: in particular, when  $G = \text{SO}(n+1)$  and  $K = \text{SO}(n) \times \text{SO}(1) = \text{SO}(n)$ , the Riemannian structure defined by the previous conventions on the  $n$ -dimensional sphere  $X = \mathbb{S}^n(\mathbb{R})$  differs from the “usual” metric<sup>2</sup> by a factor  $\sqrt{n+1}$ . But on the other hand, it will be very convenient for computations to have the same normalizations for compact Lie groups and for their symmetric spaces (we did not want to define specific conventions for each family).

<sup>2</sup>By “usual”, we mean the restriction of the standard euclidian metric of  $\mathbb{R}^{n+1}$ .



The *Laplace-Beltrami operator* on a Riemannian manifold  $M$  is the differential operator of degree 2 defined by

$$\Delta f(m) = \sum_{1 \leq i, j \leq d} g^{ij} (\nabla_{X_i} \nabla_{X_j} f(m) - \nabla_{\nabla_{X_i} X_j} f(m)),$$

where  $(X_1, \dots, X_d)$  is a basis of  $T_m M$ ,  $(g^{ij})_{i,j}$  is the inverse of the metric tensor  $(g_{ij} = \langle X_i | X_j \rangle_{T_m M})_{i,j}$ , and  $\nabla_X Y$  denotes the covariant derivative of a vector  $Y$  along a vector  $X$  and with respect to the Levi-Civita connection. In the case of a compact Lie group  $K$ , this expression can be greatly simplified as follows (see for instance [Lia04b, §2.3]). Fix once and for all an orthonormal basis  $(X_1, X_2, \dots, X_d)$  of  $\mathfrak{k}$ . On another tangent space  $T_k K$ , one transports each  $X_i$  by setting

$$X_i^l(k) = \{d_e R_k\}(X_i) \in T_k K,$$

where  $R_k$  is the multiplication on the right by  $k$ . One thus obtains a vector field  $X_i^l = \frac{\partial}{\partial x_i}$  which is left-invariant by construction and right-invariant because of the  $\text{Ad}(K)$ -invariance of the scalar product on  $\mathfrak{k}$ . Then,

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}. \quad (1.7)$$

**Definition 3.** A (standard) Brownian motion on a compact Riemannian manifold  $M$  is a continuous Feller process  $(m_t)_{t \in \mathbb{R}_+}$  whose infinitesimal generator restricted to  $\mathcal{C}^2(M)$  is  $\frac{1}{2} \Delta$ .

In the following, on a compact Lie group  $K$  or a compact symmetric space  $G/K$ , we shall also assume that  $m_0 = e$  or  $m_0 = eK$  almost surely. We shall then denote  $\mu_t$  the marginal law of the process at time  $t$ , and  $p_t^K(k) = \frac{d\mu_t}{d\eta_K}(k)$  or  $p_t^X(x) = \frac{d\mu_t}{d\eta_X}(x)$  the density of  $\mu_t$  with respect to the Haar measure. General results about hypoelliptic diffusions on manifolds ensure that these densities exist for  $t > 0$  and are continuous in time and space; we shall later give explicit formulas for them (cf. Section 2).

By using the geometry of the spaces considered and the language of Lévy processes, one can give another equivalent definition of Brownian motions. The *right increments* of a random process  $(g_t)_{t \in \mathbb{R}_+}$  with values in a (compact) Lie group  $G$  are the random variables  $r_t^s = g_s^{-1} g_t$ , so  $g_t = g_s r_t^s$  for any times  $s \leq t$ . Then, a *left Lévy process* on  $G$  is a càdlàg random process such that:

- (1) For any times  $0 = t_0 \leq t_1 \leq \dots \leq t_n$ , the right increments  $r_{t_1}^{t_0}, r_{t_2}^{t_1}, \dots, r_{t_n}^{t_{n-1}}$  are independent.
- (2) For any times  $s \leq t$ , the law of  $r_t^s$  only depends on the difference  $t - s$ :  $r_{t-s}^0 \stackrel{\text{law}}{=} r_t^s$ .

Denote  $P_t$  the operator on the space  $\mathcal{C}(G)$  of continuous functions on  $G$  defined by  $(P_t f)(g) = \mathbb{E}[f(gg_t)]$ ; and  $\mu_t$  the law of  $g_t$  assuming that  $g_0 = e_G$  almost surely. For  $h \in G$ , we also denote by  $L_h$  the operator on  $\mathcal{C}(G)$  defined by  $L_h f(g) = f(hg)$ . If  $(g_t)_{t \in \mathbb{R}_+}$  is a left Lévy process on  $G$  starting at  $g_0 = e_G$ , then:

- (1) The family of operators  $(P_t)_{t \in \mathbb{R}_+}$  is a Feller semigroup that is left  $G$ -invariant, meaning that  $P_t \circ L_h = L_h \circ P_t$  for all  $h \in G$  and for all time  $t$ . Conversely, any such Feller semigroup is the group of transitions of a left Lévy process which is unique in law.
- (2) The family of laws  $(\mu_t)_{t \in \mathbb{R}_+}$  is a semigroup of probability measures for the convolution product of measures

$$(\mu * \nu)(f) = \int_{G^2} f(gh) d\mu(g) d\nu(h).$$

Hence,  $\mu_s * \mu_t = \mu_{s+t}$  for any  $s$  and  $t$ . Moreover, this semigroup is continuous, *i.e.*, the limit in law  $\lim_{t \rightarrow 0} \mu_t$  exists and is the Dirac measure  $\delta_e$ . Conversely, given such a semigroup of measures, there is always a corresponding left Lévy process, and it is unique in law.

Thus, left Lévy processes are the same as left  $G$ -invariant Feller semigroups of operators, and they are also the same as continuous semigroups of probability measures on  $G$ . In particular, on a compact Lie



group, they are characterized by their infinitesimal generator

$$Lf(g) = \lim_{t \rightarrow \infty} \frac{P_t f(g) - f(g)}{t}$$

defined on a suitable subspace of  $\mathcal{C}(G)$ . Hunt's theorem (cf. [Hun56]) then characterizes the possible infinitesimal generators of (left) Lévy processes on a Lie group. In the case of a Lévy process with continuous paths, it can be stated as follows:

**Theorem 4** (Hunt). *The domain of the infinitesimal generator of a continuous left Lévy process traced on a compact Lie group  $G$  contains  $\mathcal{C}^2(G)$ , and it writes uniquely as*

$$L = \frac{1}{2} \sum_{i,j=1}^d a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d c^i \frac{\partial}{\partial x_i}, \quad (1.8)$$

where the diffusion part  $(a^{ij})_{1 \leq i,j \leq d}$  is a non-negative symmetric matrix, and the drift part  $(c^i)_{1 \leq i \leq d}$  is any vector.

Suppose then that  $(g_t)_{t \in \mathbb{R}_+}$  is a continuous Lévy process on a simple compact Lie group  $G$ , starting from  $e$  and with the additional property that  $(hg_t h^{-1})_{t \in \mathbb{R}_+}$  and  $(g_t)_{t \in \mathbb{R}_+}$  have the same law in  $\mathcal{C}(\mathbb{R}_+, G)$  for every  $h$ . These hypotheses imply in Equation (1.8) that  $(a^{ij})_{i,j} = (a\delta_{ij})_{i,j}$  for some constant  $a > 0$ , and that  $(c^i)_i = 0$ . Thus, on a simple compact Lie group  $K$ , up to a linear change of time  $t \mapsto at$ , a conjugacy-invariant continuous left Lévy process is a Brownian motion in the sense of Definition 3. Similarly, on a simple compact symmetric space  $G/K$ , up to a linear change of time, the image  $(g_t K)_{t \in \mathbb{R}_+}$  of a conjugacy-invariant continuous left Lévy process on  $G$  is a Brownian motion in the sense of Definition 3. This second definition of Brownian motions on compact symmetric spaces has the following important consequence:

**Lemma 5.** *Let  $\mu_t$  be the law of a Brownian motion on a compact Lie group  $K$  or on a compact symmetric space  $G/K$ . The total variation distance  $d_{TV}(\mu_t, \text{Haar})$  is a non-increasing function of  $t$ .*

*Proof.* First, let us treat the case of compact Lie groups. We define the convolution product of two functions on  $K$  by

$$(f_1 * f_2)(k) = \int_K f_1(h) f_2(h^{-1}k) dh.$$

If  $f_1, f_2$  are in  $\mathcal{L}^1(K, d\eta_K)$ , then  $f_1 * f_2$  is again in  $\mathcal{L}^1(K)$ , with  $\|f_1 * f_2\|_{\mathcal{L}^1(K)} \leq \|f_1\|_{\mathcal{L}^1(K)} \|f_2\|_{\mathcal{L}^1(K)}$ . Now, since  $\mu_{s+t} = \mu_s * \mu_t$ , the densities of the Brownian motion also satisfy  $p_{s+t}^K = p_s^K * p_t^K$ . Consequently,

$$\begin{aligned} 2 d_{TV}(\mu_{s+t}, \eta_K) &= \|p_{s+t}^K - 1\|_{\mathcal{L}^1(K)} = \|(p_s^K - 1) * p_t^K\|_{\mathcal{L}^1(K)} \\ &\leq \|p_s^K - 1\|_{\mathcal{L}^1(K)} \|p_t^K\|_{\mathcal{L}^1(K)} = \|p_s^K - 1\|_{\mathcal{L}^1(G)} = 2 d_{TV}(\mu_s, \eta_K). \end{aligned}$$

The proof is thus done in the group case. For a compact symmetric space  $X = G/K$ , denote  $p_t^G$  the density of the Brownian motion on  $G$ , and  $p_t^X$  the density of the Brownian motion on  $X$ . Since the Brownian motion on  $X$  is the image of the Brownian motion on  $G$  by  $\pi : G \rightarrow G/K$ , one has:

$$\forall x = gK, \quad p_t^X(x) = \int_K p_t^G(gk) dk.$$

As a consequence,

$$\begin{aligned} \|p_{s+t}^X - 1\|_{\mathcal{L}^1(X)} &= \int_G |p_{s+t}^X(gK) - 1| dg = \int_G \left| \int_K (p_{s+t}^G(gk) - 1) dk \right| dg \\ &= \int_G \left| \int_{K \times G} (p_s^G(h^{-1}gk) - 1) p_t^G(h) dk dh \right| dg = \int_G \left| \int_G (p_s^X(h^{-1}gK) - 1) p_t^G(h) dh \right| dg \\ &\leq \int_{G \times G} |p_s^X(h^{-1}gK) - 1| |p_t^G(h)| dh dg = \|p_s^X - 1\|_{\mathcal{L}^1(X)} \|p_t^G\|_{\mathcal{L}^1(G)} = \|p_s^X - 1\|_{\mathcal{L}^1(X)}, \end{aligned}$$

so  $d_{\text{TV}}(\mu_{s+t}, \eta_X) \leq d_{\text{TV}}(\mu_s, \eta_X)$  also in the case of symmetric spaces. Notice that our proof works for any projection on a homogeneous space  $G/K$  of a Lévy process (not necessarily continuous) traced on a compact Lie group  $G$ . Later, this property will allow us to compute estimates of  $d_{\text{TV}}(\mu_t, \eta_X)$  only for  $t$  around the cut-off time. Indeed, if one has for instance an (exponentially small) estimate of  $1 - d_{\text{TV}}(\mu_{t_0}, \eta_X)$  at time  $t_0 = (1 - \varepsilon) t_{\text{cut-off}}$ , then the same estimate will also hold for  $1 - d_{\text{TV}}(\mu_t, \eta_X)$  with  $t < t_0$ .  $\square$

*Remark.* Actually, the same result holds for the  $\mathcal{L}^p$ -norm of  $p_t(x) - 1$ , and in the broader setting of Markov processes with a stationary measure; see *e.g.* [CSC08, Proposition 3.1]. Our proof is more elementary since it only uses the triangular inequality  $|\int_X f(x) dx| \leq \int_X |f(x)| dx$  and algebraic manipulations with the semigroup of measures.

A third equivalent definition of Brownian motions on compact Lie groups is by mean of stochastic differential equations. More precisely, given a Brownian motion  $(k_t)_{t \in \mathbb{R}_+}$  traced on a compact Lie group  $K$ , there exists a (trajectorially unique) standard  $d$ -dimensional Brownian motion  $(W_t)_{t \in \mathbb{R}_+}$  on the Lie algebra  $\mathfrak{k} = \text{Vect}(X_1, X_2, \dots, X_d)$  such that, for every function  $f \in \mathcal{C}^2(K)$ , the following stochastic differential equation holds:

$$f(k_t) = f(k_0) + \sum_{i=1}^d \int_{s=0}^t \frac{\partial f(k_s)}{\partial x_i} \circ dW_t^i = f(g_0) + \sum_{i=1}^d \int_{s=0}^t \frac{\partial f(k_s)}{\partial x_i} \cdot dW_t^i + \frac{1}{2} \sum_{i=1}^d \int_{s=0}^t \frac{\partial^2 f(k_s)}{\partial x_i^2} ds,$$

where the symbol  $\circ$  corresponds to Stratonovitch's stochastic integral, and the symbol  $\cdot$  corresponds to Itô's stochastic integral. So for instance, on a unitary group  $U(n, \mathbb{C})$ , the Brownian motion is the solution of the SDE

$$U_0 = I_n \quad ; \quad dU_t = iU_t \cdot dH_t - \frac{1}{2} U_t dt,$$

where  $(H_t)_{t \in \mathbb{R}_+}$  is a Brownian hermitian matrix normalized so that at time  $t = 1$  the diagonal coefficients are independent real gaussian variables of variance  $1/n$ , and the upper-diagonal coefficients are independent complex gaussian variables with real and imaginary parts independent and of variance  $1/2n$ . Similarly, on  $SO(n, \mathbb{R})$ , the Brownian motion is the solution of the SDE

$$O_0 = I_n \quad ; \quad dO_t = O_t \cdot dA_t - \frac{n-1}{2n} O_t dt,$$

where  $(A_t)_{t \in \mathbb{R}_+}$  is an skew-symmetric Brownian matrix normalized so that at time  $t = 1$  the upper-diagonal coefficients are independent real gaussian variables of variance  $1/n$ . In the general case, let us introduce the *Casimir operator*

$$C = \sum_{i=1}^d X_i \otimes X_i. \tag{1.9}$$

This tensor should be considered as an element of the universal enveloping algebra  $U(\mathfrak{k})$ . Then, for every representation  $\pi : K \rightarrow \text{GL}(V)$ , the image of  $C$  by the infinitesimal representation  $d\pi : U(\mathfrak{k}) \rightarrow \text{End}(V)$  commutes with  $d\pi(\mathfrak{g})$ . In particular, for an irreducible representation  $V$ ,  $d\pi(C)$  is a scalar multiple  $\kappa_V \text{id}_V$  of  $\text{id}_V$ . Suppose  $K$  is a classical simple Lie group. Then its “geometric” representation is irreducible, so  $\sum_{i=1}^d X_i^2 = \alpha_{\mathfrak{g}} I_n$  if one sees the  $X_i$ 's as matrices in  $M(n, \mathbb{R})$  or  $M(n, \mathbb{C})$  or  $M(n, \mathbb{H})$ . The stochastic differential equation satisfied by a Brownian motion on  $K$  is then

$$k_0 = e_K \quad ; \quad dk_t = k_t dB_t + \frac{\alpha_{\mathfrak{g}}}{2} k_t dt,$$

where  $B_t = \sum_{i=1}^d W_t^i X_i$  is a standard Brownian motion on the Lie algebra  $\mathfrak{k}$ . The constant  $\alpha_{\mathfrak{g}}$  is given in the classical cases by

$$\alpha_{\mathfrak{su}(n)} = -\frac{n^2 - 1}{n^2} \quad ; \quad \alpha_{\mathfrak{so}(n)} = -\frac{n - 1}{n} \quad ; \quad \alpha_{\mathfrak{sp}(n)} = -\frac{2n + 1}{2n}$$

see [Lév11, Lemma 1.2]. These Casimir operators will play a prominent role in the computation of the densities of these Brownian motions (*cf.* §2.2), and also at the end of this paper (§4.1), see Lemma 27.

**1.4. Chen-Saloff-Coste results on  $\mathcal{L}^p$ -cut-offs of Markov processes.** Fix  $p \in [1, \infty)$ , and consider a Markov process  $\mathfrak{X} = (x_t)_{t \in \mathbb{R}_+}$  with values in a measurable space  $(X, \mathcal{B}(X))$ , and admitting an invariant probability  $\eta$ . One denotes  $\mu_{t,x}$  the marginal law of  $x_t$  assuming  $x_0 = x$  almost surely, and

$$d_t^p(\mathfrak{X}) = \max_{x \in X} \left( \int_X \left| \frac{d\mu_{t,x}}{d\eta}(y) - 1 \right|^p \eta(dy) \right)^{\frac{1}{p}},$$

with by convention

$$d_t^p(\mathfrak{X}) = \begin{cases} 2 & \text{if } p = 1, \\ +\infty & \text{if } p > 1, \end{cases}$$

when  $\mu_{t,x}$  is not absolutely continuous with respect to  $\eta$ . This is obviously a generalization of the total variation distance to the stationary measure. In virtue of the remark stated just after Lemma 5,  $t \mapsto d_t^p(\mathfrak{X})$  is always non-increasing. A sequence of Markov processes  $(\mathfrak{X}^{(n)})_{n \in \mathbb{N}}$  with values in measurable spaces  $(X^{(n)}, \mathcal{B}(X^{(n)}))_{n \in \mathbb{N}}$  is said to have a *max- $\mathcal{L}^p$ -cut-off* with cut-off times  $(t^{(n)})_{n \in \mathbb{N}}$  if

$$\lim_{n \rightarrow \infty} \left( \sup_{t > (1+\varepsilon)t^{(n)}} d_t^p(\mathfrak{X}^{(n)}) \right) = 0 \quad ; \quad \lim_{n \rightarrow \infty} \left( \inf_{t < (1-\varepsilon)t^{(n)}} d_t^p(\mathfrak{X}^{(n)}) \right) = \limsup_{n \rightarrow \infty} d_0^p(\mathfrak{X}^{(n)}) = M > 0$$

for every  $\varepsilon \in (0, 1)$  — usually  $M$  will be equal to 2 or  $+\infty$ . A generalization of Theorem 1 ensures that these  $\mathcal{L}^p$ -cut-offs occur for instance in the case of riffle shuffles of cards, with  $t^{(n)} = \frac{3 \log n}{2 \log 2}$  for every  $p \in [1, +\infty)$ .

In [CSC08], Chen and Saloff-Coste shown that a general criterion due to Peres ensures a  $\mathcal{L}^{p>1}$ -cut-off for a sequence of Markov processes; but then one does not know necessarily the value of the cut-off time  $t^{(n)}$ . Call *spectral gap*  $\lambda(\mathfrak{X})$  of a Markov process  $\mathfrak{X}$  the largest  $c \geq 0$  such that for all  $f \in \mathcal{L}^2(X, \eta)$  and all time  $t$ ,  $\|(P_t - \eta)f\|_{\mathcal{L}^2(X)} \leq e^{-tc} \|f\|_{\mathcal{L}^2(X)}$ , where  $(P_t)_{t \in \mathbb{R}_+}$  stands for the semigroup associated to the Markov process. For simplicity, we shall assume that these operators  $P_t$  are normal; this in particular the case for Brownian motions on compact Lie groups or symmetric spaces.

**Theorem 6** (Chen-Saloff-Coste). *Fix  $p \in (1, \infty)$ . One considers a family of normal Markov processes  $(\mathfrak{X}^{(n)})_{n \in \mathbb{N}}$  with spectral gaps  $\lambda^{(n)}$ , and one assumes that  $\lim_{t \rightarrow \infty} d_t^p(\mathfrak{X}^{(n)}) = 0$  for every  $n$ . For  $\varepsilon_0 > 0$  fixed, set*

$$t^{(n)} = \inf\{t : d_t^p(\mathfrak{X}^{(n)}) \leq \varepsilon_0\}.$$

*The family of Markov processes has a max- $\mathcal{L}^p$ -cut-off if and only if Peres' criterion is satisfied:*

$$\lim_{n \rightarrow \infty} \lambda^{(n)} t^{(n)} = +\infty.$$

*In this case, the sequence  $(t^{(n)})_{n \in \mathbb{N}}$  gives the values of the cut-off times.*

Notice that a lower bound on  $t^{(n)}$  also ensures the cut-off phenomenon; but in this case, the cut-off time remains *unknown*. Nevertheless, an important application of this general criterion is the following result (see [CSC08, Theorem 1.2], and also [SC04, Theorem 1.1 and 1.2]):

**Corollary 7** (Saloff-Coste). *Consider the Brownian motions traced on the special orthogonal groups  $\text{SO}(n, \mathbb{R})$ , with the normalization of the metric detailed in the previous paragraph. They exhibit for every  $p \in (1, \infty)$  a cut-off with  $t^{(n)}$  asymptotically between  $2 \log n$  and  $4 \log n$  — notice that  $t^{(n)}$  depends on  $p$ .*

Indeed, the spectral gap stays bounded and has a non-negative limit (which we shall compute later), whereas  $t^{(n)}$  was shown by Saloff-Coste to be a  $O(\log n)$ . Similar results are presented in [SC04] in the broader setting of simple compact Lie groups or compact symmetric spaces, but without a proof of the cut-off phenomenon (Saloff-Coste gave a window for  $t^{(n)}$  for every  $p \in [1, +\infty]$ ). The main result of our paper is that a cut-off indeed occurs for every  $p \in [1, +\infty)$ , for every classical simple compact Lie group or classical simple compact symmetric space, and with a cut-off time equal to  $\log n$  or  $2 \log n$ .

depending on the type of the space considered. In particular, the main improvements in comparison to the aforementioned theorems are the followings:

- (1) the case  $p = 1$  is now included;
- (2) one knows the precise value of the cut-off time.

To obtain this result, we more or less had to *compute*  $t^{(n)}$ . Hence this paper is mostly a case-by-case analysis, with precise explicit estimates and a few computational tricks (in particular we depart a lot from the generality of Theorem 6).

**1.5. Statement of the main results and discriminating events.** The precise statement of our main result is the following:

**Theorem 8.** *Let  $\mu_t$  be the marginal law of the Brownian motion traced on a classical simple compact Lie group, or on classical simple compact symmetric space. There exists positive constants  $\alpha$ ,  $\gamma_b$ ,  $\gamma_a$ ,  $c$ ,  $C$  and an integer  $n_0$  such that in each family, for all  $n \geq n_0$ ,*

$$\forall \varepsilon \in (0, 1/4), \quad d_{\text{TV}}(\mu_t, \text{Haar}) \geq 1 - \frac{c}{n^{\gamma_b \varepsilon}} \quad \text{if } t = \alpha(1 - \varepsilon) \log n; \quad (1.10)$$

$$\forall \varepsilon \in (0, \infty), \quad d_{\text{TV}}(\mu_t, \text{Haar}) \leq \frac{C}{n^{\gamma_a \varepsilon/4}} \quad \text{if } t = \alpha(1 + \varepsilon) \log n. \quad (1.11)$$

The constants  $\alpha$ ,  $\gamma_b$  and  $\gamma_a$  are determined by the type of the space considered, and then one can make the following choices for  $n_0$ ,  $c$  and  $C$ :

$K$ or $G/K$	$\beta$	$\alpha$	$\gamma_b$	$\gamma_a$	$n_0$	$c$	$C$
$\text{SO}(n, \mathbb{R})$	1	2	2	2	10	36	6
$\text{SU}(n, \mathbb{C})$	2	2	2	4	2	8	10
$\text{USp}(n, \mathbb{H})$	4	2	2	2	3	5	3
$\text{Gr}(n, q, \mathbb{R})$	1	1	1	1	10	32	2
$\text{Gr}(n, q, \mathbb{C})$	2	1	1	2	2	32	2
$\text{Gr}(n, q, \mathbb{H})$	4	1	1	1	3	16	2
$\text{SO}(2n, \mathbb{R})/\text{U}(n, \mathbb{C})$	1	1	2	1	10	8	2
$\text{SU}(n, \mathbb{C})/\text{SO}(n, \mathbb{R})$	2	1	2	2	2	24	8
$\text{SU}(2n, \mathbb{C})/\text{USp}(n, \mathbb{H})$	2	1	2	2	2	22	8
$\text{USp}(n, \mathbb{H})/\text{U}(n, \mathbb{C})$	4	1	2	1	3	17	2

As the function  $t \mapsto d_{\text{TV}}(\mu_t, \text{Haar})$  is non-increasing in  $t$ , the aspect of this function in the scale  $t \propto \log n$  is then always as on Figure 1.

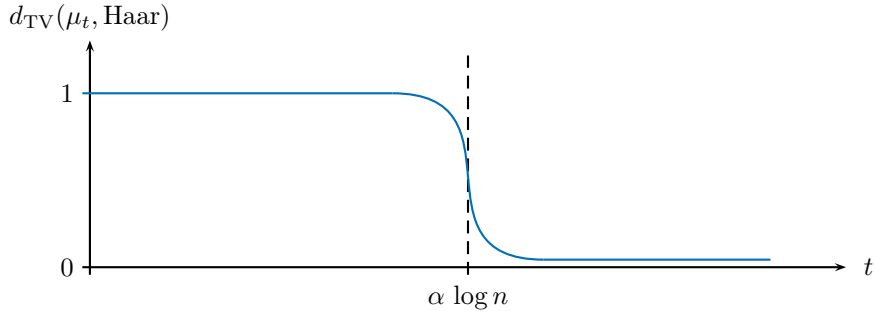


FIGURE 1. Aspect of the function  $t \mapsto d_{\text{TV}}(\mu_t, \text{Haar})$  for the Brownian motion on a classical simple compact Lie group or on a classical simple compact symmetric space.

The constants  $c$  and  $C$  in Theorem 8 can be slightly improved by raising the integer  $n_0$ ; the restriction  $n \geq n_0$  will only be used to ease certain computations and to get reasonable constants  $c$  and  $C$ . A result similar to Theorem 8 has been proved by Rosenthal and Porod in [Ros94, Por96a, Por96b] for random products of (real, or complex, or quaternionic) reflections. Our proofs are really inspired by their proofs, though quite different in the details of the computations. For the upper bound (1.11), it has long been known that if  $\lambda(X_n)$  denotes the spectral gap of the heat semigroup associated to the infinitesimal generator  $L = \frac{1}{2}\Delta$ , then for  $n$  fixed, the total variation distance  $d_{\text{TV}}(\mu_t, \eta_{X_n})$  decreases exponentially fast:

$$d_{\text{TV}}(\mu_t, \eta_{X_n}) \leq C(X_n) e^{-\lambda(X_n)t}.$$

We refer for instance to [Lia04a], where such a bound is proved for any “non-degenerate” Lévy process on a compact Lie group. Consider now the family of spaces  $(X_n)_{n \in \mathbb{N}}$ , and suppose that  $C(X_n) = C n^\delta$ , and that  $\lambda(X_n)$  stays almost constant to  $\lambda$  — this last condition is ensured by the normalization (1.6). Then, one obtains for  $t = (1 + \varepsilon) \frac{\delta}{\lambda} \log n$  the bound

$$d_{\text{TV}}(\mu_t, \eta_{X_n}) \leq \frac{C}{n^{\delta\varepsilon}}.$$

Thus in theory, the upper bound (1.11) follows from the calculations of the constants  $C(X_n)$  and  $\lambda(X_n)$  in each classical family. It is very hard to find directly a constant  $C(X_n)$  that works for every time  $t$ . But on the other side, by using the representation theory of the classical simple compact Lie groups (cf. Section 2), one can determine series of negative exponentials that dominates the total variation distance; see Proposition 16. In these series, the “least negative” exponentials give the correct order of decay  $\lambda(X_n)$ . It remains then to prove that the other terms can be uniformly bounded. This is tedious, but doable, and these precise estimates are shown in Section 3: we shall adapt and improve the arguments of [Ros94, Por96a, Por96b, CSST08].

As for the lower bound (1.10), it is obtained by looking at *discriminating events*, that have a probability close to 1 with respect to a marginal law  $\mu_t$  with  $t < t_{\text{cut-off}}$ , and close to 0 with respect to the Haar measure. In the case of riffle shuffles, the sizes of the *runs* of a permutation enable one to discriminate a random shuffle of order  $k < k_{\text{cut-off}}$  from a uniform permutation. Indeed, call run of a permutation  $w = w_1 w_2 \dots w_n$  a maximal<sup>3</sup> subsequence  $w_{i_1} w_{i_2} \dots w_{i_r}$  with  $i_1 < i_2 < \dots < i_r$  and  $w_{i_j+1} = w_{i_j} + 1$  for every  $j$ . For example, the permutation 17283459X6 has two runs, namely, 123456 and 789X. Now, consider for instance a random permutation  $\sigma$  of size  $n = 52$  obtained by shuffling 3 times the deck. Each shuffle splits a run into at most 2 runs, so  $\sigma$  has at most  $8 = 2^3$  runs, and the probability that one of them is of length 1 is extremely small. On the contrary, a random uniform permutation has with probability close to 1 at least one run of length 1. Indeed,

$$\sigma \text{ has a run of length 1} \iff \exists i \in \llbracket 2, n-1 \rrbracket, \sigma^{-1}(i-1) > \sigma^{-1}(i) > \sigma^{-1}(i+1).$$

For a given  $i \in \llbracket 2, n-1 \rrbracket$ , the probability under  $\mathbb{U}$  of the event  $A_i = \{\sigma^{-1}(i-1) > \sigma^{-1}(i) > \sigma^{-1}(i+1)\}$  is  $\frac{1}{6}$ . Moreover, if  $|i-j| \geq 3$ , then  $A_i$  and  $A_j$  are independent under  $\mathbb{U}$ . As a consequence,

$$\mathbb{U}[\sigma \text{ has a run of length 1}] \geq 1 - \mathbb{U}\left[\bigcap_{\substack{i \in \llbracket 2, n-1 \rrbracket \\ i=3k+2}} A_i^c\right] = 1 - \left(\frac{5}{6}\right)^{\lfloor \frac{n}{3} \rfloor}.$$

Thus, the event  $A = \{\sigma \text{ has a run of length 1}\}$  separates the uniform measure  $\mathbb{U}$  from the measure  $\mathbb{P}^{(3)}$ , and by improving this argument, one can show similar inequalities for  $\mathbb{P}^{(k)}$  with  $k < k_{\text{cut-off}}$ .

Now, in the case of a Brownian motion on a classical compact Lie group, this is the *trace* of the matrices that allows to discriminate Haar distributed elements and random Brownian elements before cut-off time. Indeed, consider for instance a random unitary matrix  $U_n$  of size  $n$ , taken under the Haar

<sup>3</sup>By “maximal”, we mean that one cannot find a larger subsequence in the sense of the inclusion; however, runs may have distinct lengths.

measure or under the marginal law  $\mu_t$  of the Brownian motion at a given time  $t$ . Then,  $\text{tr } U_n$  is a complex valued random variable, and we shall see that

$$\mathbb{E} [|\text{tr } U_n - m|^2] \leq 1,$$

where  $m$  is the mean of  $\text{tr } U_n$ ; and this, for any  $n \geq 1$  and any time  $t \geq 0$  if  $U_n \sim \mu_t$ . However,  $m = 0$  under the Haar measure, whereas  $|m| \gg 1$  for  $t < t_{\text{cut-off}}$ . So, the trace of a Brownian unitary matrix before cut-off time will never “look the same” as the trace of an Haar distributed unitary matrix. Up to

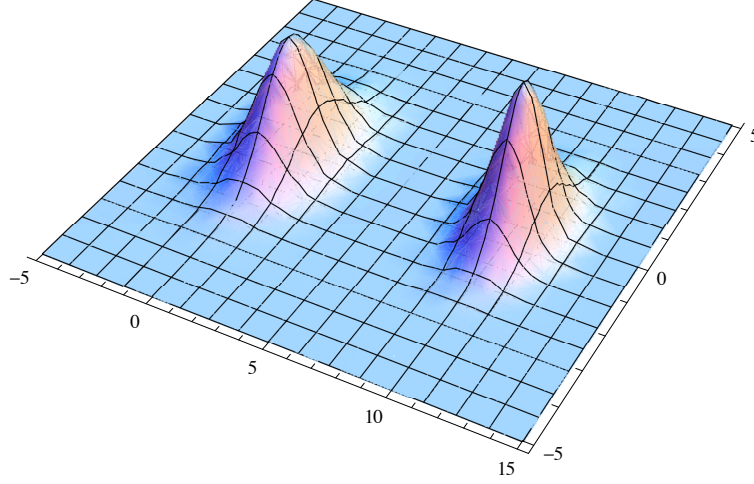


FIGURE 2. Density of the trace  $\text{tr } U_n$  of a random unitary matrix, with  $U_n \sim \text{Haar}$  for the left peak, and  $U_n \sim \mu_{t < t_{\text{cut-off}}}$  for the right peak.

a minor modification, the same argument will work for special orthogonal groups and compact special orthogonal groups — in this later case, the trace of a quaternionic matrix of size  $n$  is defined as the trace of the corresponding complex matrix of size  $2n$ , cf. the remark at the end of §1.2.

Over the classical simple compact symmetric spaces, the trace of matrices will be replaced by a zonal spherical function “of minimal non-zero weight”; these minimal zonal spherical functions are also those that give the order of decay of the series of negative exponentials that dominate  $d_{\text{TV}}(\mu_t, \text{Haar})$  after the cut-off time. This argument for the lower bound was already known, since it has been used successfully in [SC94] to prove the cut-off phenomenon over spheres: we have simply extended it to the case of general simple compact symmetric spaces (cf. Section 4).

An important consequence of Theorem 8 and its proof is that one also has a  $\max\text{-}\mathcal{L}^p$ -cut-off for every  $p \in [1, \infty]$ . Moreover, the value of the cut-off time is known when  $p \in [1, 2]$ . More precisely:

**Corollary 9.** *For every  $p \in [1, +\infty]$ , the family of Brownian motions  $(\mathfrak{X}_n)_{n \in \mathbb{N}}$  traced on simple compact Lie groups  $(K_n)_{n \in \mathbb{N}}$  in one of the three classical families (respectively, on simple compact symmetric spaces of type non-group  $(X_n)_{n \in \mathbb{N}}$  in one of the seven classical families) has a  $\max\text{-}\mathcal{L}^p$ -cut-off. If  $p \in [1, 2]$ , it is with respect to the sequence  $t^{(n)} = 2 \log n$  (respectively,  $t^{(n)} = \log n$ ).*

*Proof.* The upper bound in Theorem 8 will be shown by using Cauchy-Schwarz inequality and estimating the  $\mathcal{L}^2$ -norm of  $\left| \frac{d\mu_t}{d\eta_{X_n}} - 1 \right|$ , which can be written as a series  $S_n(t)$  of negative exponentials. Section 3 is devoted to the proof of the fact that  $S_n(t)$  is small after cut-off time, and on the other hand, the same series trivially goes to infinity before cut-off time, because some of its terms go to infinity (consider for instance the term indexed by the “minimal” label identified in Lemma 17). Thus, our proof of Theorem 8 implies readily a  $\mathcal{L}^2$ -cut-off; and since the Brownian motion is invariant by action of the isometry

group, it is even a  $\max\text{-}\mathcal{L}^2$ -cut-off. We can then use [CSC08, Theorem 5.3] to obtain the existence of a  $\max\text{-}\mathcal{L}^p$ -cut-off for every  $p \in (1, +\infty]$ , and the comparison theorem of mixing times [CSC08, Proposition 5.1] to get the value of the cut-off time when  $p$  is between 1 and 2. When  $p = +\infty$ , [CSC08, Theorem 5.3] also gives the value of the cut-off time: it is  $4 \log n$  in the group case, and  $2 \log n$  in the non-group case. However, when  $p \in (2, +\infty)$ , one still does not know the value of the mixing time: one has only the window  $\alpha \log n \leq t^{(n)} \leq 2\alpha \log n$ .  $\square$

**Organization of the paper.** In Section 2, we recall the basics of representation theory and harmonic analysis on compact symmetric spaces, with a particular emphasis on explicit formulas since we will need them in each case. All of it is really classical and of course well-known by the experts. Nevertheless, in order to be as self-contained as possible, we have decided to include in our paper some details of the proof of the totally explicit formulas appearing in Theorem 15. It also enables us to fix the notations related to the harmonic analysis of the classical compact Lie groups and compact symmetric spaces. In Section 3, we use the explicit expansion of the densities to establish precise upper bounds on  $\|p_t - 1\|_{\mathcal{L}^2(X, \eta)}$ ; by Cauchy-Schwarz we obtain similar upper bounds on  $d_{TV}(\mu_t, \eta)$ . The main idea is to control the growth of the dimension of an irreducible spherical representation involved in the expansion of  $p_t$  when the corresponding highest weight grows in the lattice of weights (§3.2). The crucial fact, which was apparently unknown, is that precisely at cut-off time, the quantity

$$\begin{cases} (D^\lambda)^2 e^{-t_{\text{cut-off}} B_n(\lambda)} & \text{in the group case,} \\ D^\lambda e^{-t_{\text{cut-off}} B_n(\lambda)} & \text{in the non-group case,} \end{cases}$$

stays *bounded* for every  $n$  and every  $\lambda$ ;  $D^\lambda$  being the dimension of the irreducible or spherical irreducible representation of label  $\lambda$ , and  $-B_n(\lambda)$  the associated eigenvalue of the Laplace-Beltrami operator. Combining this argument with a simple analysis of the generating series

$$\sum_{\lambda \text{ partition}} x^{|\lambda|} = \prod_{i \geq 1} \frac{1}{1 - x^i},$$

this is sufficient to get a correct upper bound after cut-off time.

Section 4 is then devoted to the proof of the lower bounds. We use in each case a “minimal” zonal spherical function (the trace of matrices in the case of groups; see §4.1), and we compute its expectation and variance under Haar measure and Brownian measures (§4.2). A simple application of Bienaymé-Chebyshev’s inequality will then show that the chosen zonal spherical function is indeed discriminating. An algebraic difficulty occurs in the case of symmetric spaces  $G/K$  of type non-group, as one has to compute the expansion in zonal functions of the square of the discriminating zonal function, and this is far less obvious than in the case of irreducible characters. The problem is solved by writing the discriminating zonal function in terms of the coefficients of the matrices in the isometry group  $G$ , and by computing the joint moments of these coefficients under a Brownian measure. The combinations of negative exponentials appearing in these formulas are then in correspondence with the expansions of the squares of the discriminating zonal spherical functions.

**Acknowledgements.** Many thanks are due to Yacine Barhoumi, Philippe Biane, Florent Benaych-Georges, Reda Chhaïbi, Djalil Chafaï, Kenneth Maples, Ashkan Nikeghbali and Simon Pépin-Lehalleur for discussions around the cut-off phenomenon and the representation theory of Lie groups.



## 2. FOURIER EXPANSION OF THE DENSITIES

In this section, we explain how to compute the density  $p_t^K(k)$  or  $p_t^X(x)$  of the marginal law  $\mu_t$  of the Brownian motion traced on a compact Lie group  $K$  or a compact symmetric space  $X = G/K$ . This computation is done in an abstract setting for instance in [Lia04a] or [App11], and we shall give at the end of this section its concrete counterpart in each classical case, see Theorem 15. The main ingredients of the computation are:

- (1) Peter-Weyl's theorem and its refinement due to Cartan, that ensures that the matrix coefficients of the irreducible representations of  $K$  (respectively, of the irreducible spherical representations of  $G$ ) form an orthogonal basis of  $\mathcal{L}^2(K, \eta)$  (respectively, of  $\mathcal{L}^2(G/K, \eta)$ ), and that the irreducible characters (respectively, the zonal spherical functions) form an orthogonal basis of  $\mathcal{L}^2(K, \eta)^K$  (respectively, of  $\mathcal{L}^2(G/K, \eta)^K$ ); see §2.1.
- (2) the classical highest weight theory, that describes the irreducible representations of a compact simple Lie group and give formulas for their dimensions and characters; see §2.2.

On these subjects, we refer to the two books by Helgason [Hel78, Hel84], and also to [BD85, Var89, FH91, Far08, GW09] for the representation theory of compact Lie groups. We shall only recall what is needed in order to have a good understanding of the formulas of Theorem 15.

**2.1. Peter-Weyl's theorem and Cartan's refinement.** Peter-Weyl's theorem deals with the extension to arbitrary (possibly non-commutative) compact topological groups of the well-known Fourier theory for functions on the circle  $\mathbb{T}$ . Namely, if  $f \in \mathcal{L}^2(\mathbb{T})$ , then of course

$$f(\theta) = \sum_{n \in \mathbb{Z}} \tilde{f}(n) e^{ni\theta} \quad ; \quad \|f\|_{\mathcal{L}^2(\mathbb{T})}^2 = \sum_{n \in \mathbb{Z}} |\tilde{f}(n)|^2$$

where  $\tilde{f}(n) = \int_0^{2\pi} f(e^{-i\theta}) e^{ni\theta} \frac{d\theta}{2\pi}$  is the  $n$ -th Fourier coefficient of  $f$ .

Let  $K$  be a compact (Lie) group, and  $\widehat{K}$  be the set of isomorphism classes of irreducible complex linear representations of  $K$ . Each class  $\lambda \in \widehat{K}$  is finite-dimensional, and we shall denote  $V^\lambda$  the corresponding complex vector space;  $\rho^\lambda : K \rightarrow \text{U}(V^\lambda)$  the representation morphism<sup>4</sup>;  $D^\lambda = \dim_{\mathbb{C}} V^\lambda$  the dimension of the representation;  $\chi^\lambda(\cdot) = \text{tr} \rho^\lambda(\cdot)$  the character; and  $\widehat{\chi}^\lambda(\cdot) = \chi^\lambda(\cdot)/D^\lambda$  the normalized character. An Hermitian scalar product on  $\text{End}(V^\lambda)$  is  $\langle M | N \rangle_{\text{End}(V^\lambda)} = D^\lambda \text{tr}(M^\dagger N)$ . For every class  $\lambda$  and every function  $f \in \mathcal{L}^2(K)$ , we set

$$\widehat{f}(\lambda) = \int_K f(k) \rho^\lambda(k) dk;$$

this is an element of  $\text{End}(V^\lambda)$ . We refer to [BD85, Far08] for a proof of the following results.

**Theorem 10** (Peter-Weyl). *The (non-commutative) Fourier transform*

$$\begin{aligned} \mathcal{F} : \mathcal{L}^2(K, \eta) &\rightarrow \bigoplus_{\lambda \in \widehat{K}} \text{End}(V^\lambda) \\ f &\mapsto \sum_{\lambda \in \widehat{K}} \widehat{f}(\lambda) \end{aligned}$$

*realizes an isometry and an isomorphism of (non-unital) algebras, where  $\mathcal{L}^2(K)$  is endowed with its usual scalar product and the convolution product, and  $\bigoplus_{\lambda} \text{End}(V^\lambda)$ , the Hilbertian completion of the algebraic*

<sup>4</sup>One can always endow  $V^\lambda$  with an Hermitian scalar product such that  $K$  acts by unitary transformations.

direct sum, is endowed with the Hilbertian sum of the scalar products  $\langle \cdot | \cdot \rangle_{\text{End}(V^\lambda)}$ . So, if  $f \in \mathcal{L}^2(K)$ , then

$$f(k) = \sum_{\lambda \in \widehat{K}} D^\lambda \text{tr} \left( \widehat{f}(\lambda) \rho^\lambda(k) \right) = \sum_{\lambda \in \widehat{K}} D^\lambda \text{tr} \left( \int_K f(h^{-1}k) \rho^\lambda(h) dh \right) \quad (2.1)$$

$$\|f\|_{\mathcal{L}^2(K)}^2 = \sum_{\lambda \in \widehat{K}} \left\| \widehat{f}(\lambda) \right\|_{\text{End}(V^\lambda)}^2 = \sum_{\lambda \in \widehat{K}} D^\lambda \text{tr} \left( \widehat{f}(\lambda)^\dagger \widehat{f}(\lambda) \right) \quad (2.2)$$

where  $\widehat{f}(\lambda) = \widehat{f}^-(\lambda) = \int_K f(k^{-1}) \rho^\lambda(k) dk$ .

Suppose now that  $f$  is in  $\mathcal{L}^2(K, \eta)^K$ , the subalgebra of conjugacy-invariant functions such that  $f(hkh^{-1}) = f(k)$  for every  $h, k \in K$ . Then for every class  $\lambda$ ,  $\widehat{f}(\lambda) = \widehat{\chi}^\lambda(f) \text{id}_{V^\lambda}$ , where

$$\chi^\lambda(f) = \int_K f(k) \chi^\lambda(k) dk = \text{tr} \left( \widehat{f}(\lambda) \right) \quad ; \quad \widehat{\chi}^\lambda(f) = \frac{\chi^\lambda(f)}{D^\lambda}.$$

So in this case, the Fourier expansion (2.1) and the Parseval identity (2.2) become

$$f(k) = \sum_{\lambda \in \widehat{K}} (D^\lambda)^2 \widehat{\chi}^\lambda(f) \widehat{\chi}^\lambda(k) \quad ; \quad \|f\|_{\mathcal{L}^2(K)}^2 = \sum_{\lambda \in \widehat{K}} |\chi^\lambda(f)|^2,$$

and in particular, the irreducible characters  $\chi^\lambda$  form an orthonormal basis of  $\mathcal{L}^2(K)^K$ .

Cartan gave a statement similar to Theorem 2.1, but for  $\mathcal{L}^2(G/K, \eta)$  where  $X = G/K$  is a simply connected irreducible compact symmetric space. Call *spherical* an irreducible representation  $(V^\lambda, \rho^\lambda)$  of  $G$  such that  $(V^\lambda)^K$ , the space of vectors invariant by  $\rho^\lambda(K)$ , is non-zero. Then, it is in fact one-dimensional, so one can find a vector  $e^\lambda$  of norm  $\|e^\lambda\|^2 = 1$ , unique up to multiplication by  $z \in \mathbb{T}$ , such that  $(V^\lambda)^K = \mathbb{C}e^\lambda$ . Denote then  $\mathcal{C}^\lambda(G/K)$  the set of functions from  $G$  to  $\mathbb{C}$  that can be written as

$$f(g) = f_v(g) = \langle v | \rho^\lambda(g)(e^\lambda) \rangle \quad \text{with } v \in V^\lambda. \quad (2.3)$$

Such a function is right- $K$ -invariant, so it can be considered as a function from  $G/K$  to  $\mathbb{C}$ .

**Theorem 11** (Cartan). *Let  $\widehat{G}^K$  be the set of spherical irreducible representations of  $G$ . The Hilbert space  $\mathcal{L}^2(G/K, \eta)$  is equal to the orthogonal sum  $\bigoplus_{\lambda \in \widehat{G}^K} \mathcal{C}^\lambda(G/K)$ . This decomposition corresponds to the Fourier expansion*

$$f(gK) = \sum_{\lambda \in \widehat{G}^K} D^\lambda \text{tr} \left( \int_G f(h^{-1}gK) \rho^\lambda(h) dh \right) \quad (2.4)$$

for  $f \in \mathcal{L}^2(G/K)$ .

One recovers Equation (2.1) from (2.4) by looking at symmetric spaces of type group  $(K \times K)/K$ : spherical representations in  $\widehat{G}^K$  are then the  $V^\lambda \otimes V^{\lambda'} = \text{End}(V^\lambda)$  with  $\lambda \in \widehat{K}$ , and the corresponding spherical vector is up to a scalar  $\text{id}_{V^\lambda}$ .

In each space  $\mathcal{C}^\lambda(G/K)$ , the space of left  $K$ -invariant functions is one-dimensional, and it is generated by the *zonal spherical function*

$$\phi^\lambda(gK) = \langle e^\lambda | \rho^\lambda(g)(e^\lambda) \rangle.$$

These spherical functions form an orthogonal basis of  $\mathcal{L}^2(X)^K$  when  $\lambda$  runs over  $\widehat{G}^K$ . So, a  $K$ -invariant function writes as

$$f(gK) = \sum_{\lambda \in \widehat{G}^K} D^\lambda \phi^\lambda(f^-) \phi^\lambda(gK),$$

where  $\phi^\lambda(f) = \int_{G/K} f(x) \phi^\lambda(x) dx = \langle e^\lambda | \int_G f(gK) \rho^\lambda(g)(e^\lambda) dg \rangle$ . To conclude with, notice that the decomposition given by Theorem 11 is the decomposition of  $\mathcal{L}^2(G/K, \eta)$  in common eigenspaces of the

elements of  $\mathcal{D}(G/K)$ , the commutative algebra of differential operators on  $X$  that are  $G$ -invariant. Thus, there are morphisms of algebras  $c^\lambda : \mathcal{D}(G/K) \rightarrow \mathbb{C}$  for every  $\lambda \in \widehat{G}^K$  such that

$$T(f^\lambda) = c^\lambda(T) f^\lambda$$

for every  $T \in \mathcal{D}(G/K)$  and every  $f^\lambda \in \mathcal{C}^\lambda(G/K)$ .

**2.2. Highest weight theorem and Weyl's character formula.** The theory of highest weights of representations enables us to identify  $\widehat{K}$  or  $\widehat{G}^K$ , and to compute the coefficients  $c^\lambda(L)$  associated to the Laplace-Beltrami operator. If  $G$  is a connected compact Lie group, its maximal tori are all conjugated, and every element of  $K$  is contained in a maximal torus  $T$ . Denote  $W = \text{Norm}(T)/T$  the *Weyl group* of  $G$ , and call *weight* of a representation  $V$  of  $G$  a group morphism  $\omega : T \rightarrow \mathbb{T}$  such that  $V^\omega = \{v \in V \mid \forall t \in T, t \cdot v = \omega(t) \cdot v\} \neq \emptyset$ . Every representation  $V$  of  $G$  is the direct sum of its weight subspaces  $V^\omega$ , and this decomposition is always  $W$ -invariant. Besides, the set of all weights of all representations of  $G$  is a lattice  $\mathbb{Z}\Omega$  whose rank is also the dimension of  $T$ . In the following, we fix a  $W$ -invariant scalar product<sup>5</sup> on the real vector space  $\mathbb{R}\Omega = \mathbb{Z}\Omega \otimes_{\mathbb{Z}} \mathbb{R}$ , and a closed fundamental set  $C$  for the action of the Weyl group on  $\mathbb{R}\Omega$ . We call *dominant* a weight  $\omega$  that falls in the Weyl chamber  $C$ .

A *root* of  $G$  is a non-zero weight of the adjoint representation (1.3). The set of roots  $\Phi$  is a root system, which means that certain combinatorial relations are satisfied between its elements: for instance, if  $\alpha \in \Phi$ , then  $-\alpha \in \Phi$ , and if  $\alpha, \beta \in \Phi$ , then the set of  $k \in \mathbb{Z}$  such that  $\alpha + k\beta \in \Phi$  is an interval. There is a unique way to split  $\Phi$  in a set  $\Phi_+$  of positive roots and a set  $\Phi_- = -\Phi_+$  such that

$$C = \{x \in \mathbb{R}\Omega \mid \forall \alpha \in \Phi_+, \langle x \mid \alpha \rangle \geq 0\}.$$

Call *simple* a positive root  $\alpha$  that cannot be written as the sum of two positive roots; and simple coroot an element  $\tilde{\alpha} = \frac{2\alpha}{\langle \alpha \mid \alpha \rangle}$  with  $\alpha$  simple root. Then, a distinguished basis of the lattice  $\mathbb{Z}\Omega$  is given by the *fundamental weights*  $\varpi_1, \varpi_2, \dots, \varpi_r$ , the dual basis of the basis of coroots. Hence, the sets of weights and of dominant weights have the following equivalent descriptions:

$$\begin{aligned} \mathbb{Z}\Omega &= \bigoplus_{i=1}^r \mathbb{Z}\varpi_i = \left\{ x \in \mathbb{R}\Omega \mid \forall \alpha \in \Phi, \frac{\langle x \mid \alpha \rangle}{\langle \alpha \mid \alpha \rangle} \in \mathbb{Z} \right\}; \\ \text{Dom}(\mathbb{Z}\Omega) &= \bigoplus_{i=1}^r \mathbb{N}\varpi_i = \left\{ x \in \mathbb{R}\Omega \mid \forall \alpha \in \Phi, \frac{\langle x \mid \alpha \rangle}{\langle \alpha \mid \alpha \rangle} \in \mathbb{N} \right\}. \end{aligned}$$

*Example.* The weight lattice of  $G = \text{SU}(3, \mathbb{C})$  is drawn on Figure 3;  $T = \{(x, y, z) \in \mathbb{T}^3 \mid xyz = 1\}$ ;  $\alpha_1(t) = xy^{-1}$ ,  $\alpha_2(t) = yz^{-1}$ ;  $\varpi_1(t) = x$ ,  $\varpi_2(t) = xy$ .

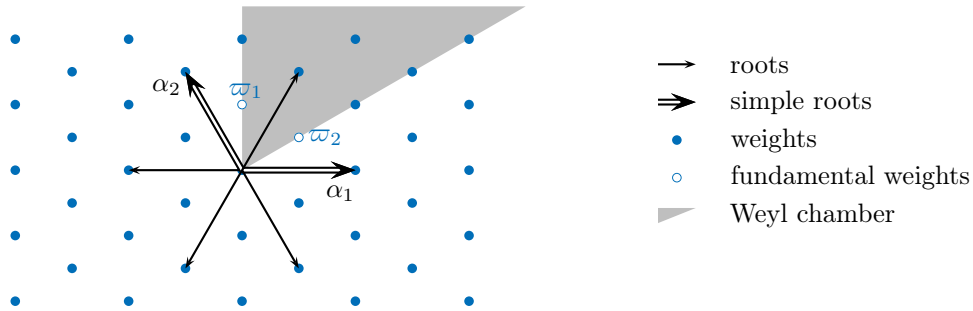


FIGURE 3. The weight lattice of type  $A_2$ , corresponding to the compact group  $\text{SU}(3, \mathbb{C})$ .

<sup>5</sup>*E.g.*, the dual of the scalar product given by Equation (1.6), where  $\mathbb{R}\Omega$  is identified with  $\mathfrak{t}^*$  by mean of  $\omega \mapsto d_e \omega$  for  $\omega \in \mathbb{Z}\Omega$ . Actually, one has to choose precisely this scalar product so that Equation (2.7) has the correct normalization.

In any case, the Weyl group  $W$  is a Coxeter group generated by the symmetries along the simple roots  $\alpha_1, \alpha_2, \dots, \alpha_r$ . In particular, it admits a signature morphism  $\varepsilon : W \rightarrow \{\pm 1\}$ .

**Theorem 12** (Weyl). *Suppose that  $G$  is a semi-simple simply connected compact Lie group, and consider the partial order induced by the convex set  $C$  on  $\mathbb{R}\Omega$ . Every irreducible representation  $V$  of  $G$  has a unique highest weight  $\omega_0$  for this order, which is then of multiplicity one and determines the isomorphism class of  $V$ . Thus, the dominant weights parametrize the irreducible representations of  $G$ . Moreover, the restriction to  $T$  of the irreducible character associated to a dominant weight  $\lambda$  is given by*

$$\chi^\lambda(t) = \frac{\sum_{\sigma \in W} \varepsilon(\sigma) \sigma(\lambda + \rho)(t)}{\sum_{\sigma \in W} \varepsilon(\sigma) \sigma(\rho)(t)}, \quad (2.5)$$

where  $\rho$  is the half-sum of all positive roots, or equivalently the sum of the fundamental weights. This formula degenerates into the dimension formula

$$D^\lambda = \dim V^\lambda = \frac{\prod_{\alpha \in \Phi_+} \langle \lambda + \rho | \alpha \rangle}{\prod_{\alpha \in \Phi_+} \langle \rho | \alpha \rangle}. \quad (2.6)$$

Notice that Equation (2.5) gives the values of the characters on any element  $g \in G$ , since they are all conjugated to an element  $t \in T$ , unique up to action of the Weyl group. Thus, Theorem 12 makes Equation (2.1) essentially explicit in the case of a conjugacy invariant function on a (semi-)simple compact Lie group  $K$ ; in particular, we shall see in a moment that the highest weights are labelled by partitions or similar combinatorial objects in all the classical cases.

The case of a compact symmetric space  $X = G/K$  of type non-group is more involved as it requires the knowledge of the involutive automorphism underlying the symmetric pair. Denote  $\theta$  an involutive automorphism of a semi-simple simply connected compact Lie group  $G$ , with  $K = G^\theta$  (see Theorem 2). Set  $P = \{g \in G \mid \theta(g) = g^{-1}\}$ ; one has then the Cartan decomposition  $G = KP$ . In addition to the assumptions of Theorem 12, one has now to suppose that the maximal torus  $T \subset G$  is chosen so that  $T^\theta = T$  and  $P \cap T$  is a maximal torus in  $P$ . This will be specially important for the harmonic analysis on Grassmannian varieties, see the next paragraph.

**Theorem 13** (Cartan-Helgason). *Under the previous assumptions, the spherical representations in  $\widehat{G}^K$  are precisely the irreducible representations in  $\widehat{G}$  that are trivial on  $K \cap T = T^\theta$ .*

This subgroup  $T^\theta$  of  $T \simeq \mathbb{T}^r$  is always the product of a subtorus  $\mathbb{T}^{s \leq r}$  with an elementary abelian 2-group  $(\mathbb{Z}/2\mathbb{Z})^t$ ; as we shall see in §2.3, this corresponds to additional conditions on the size and the parity of the parts of the partitions labeling the highest weights in  $\widehat{G}^K$  (in comparison to  $\widehat{G}$ ). The corresponding zonal spherical functions  $\phi^\lambda$  do not have in general an expression as simple as (2.5); see however [HS94, Part 1]. For most of our computations, this will not be a big problem, since we shall only use certain properties of the spherical functions — *e.g.*, their orthogonality and the formula for the dimension  $D^\lambda$  — and not their explicit form; see however §4.1.

The last ingredient in the computation of the densities is the value of the coefficient  $c^\lambda(L)$  such that

$$\frac{\Delta(f^\lambda)}{2} = c^\lambda(L) f^\lambda$$

for every function  $f^\lambda$  either in  $\mathcal{R}^\lambda(K) = \text{Vect}(\{k \mapsto (\rho^\lambda(k))_{ij}, 1 \leq i, j \leq D^\lambda\})$  in the group case, or in  $\mathcal{C}^\lambda(G/K)$  in the case of a symmetric space. In the group case, by comparing the definition of the Casimir operator (1.9) with the definition of the Laplace-Beltrami operator (1.7), one sees that  $c^\lambda(L)$  is also  $\frac{\kappa_\lambda}{2}$ , where  $\kappa_\lambda$  is the constant by which the Casimir operator  $C$  acts *via* the infinitesimal representation  $d\rho^\lambda : U(\mathfrak{k}) \rightarrow \text{End}(V^\lambda)$  — *cf.* the remark at the end of §1.3. This constant is equal to

$$\kappa_\lambda = -\langle \lambda + 2\rho \mid \lambda \rangle, \quad (2.7)$$

see [App11, Equation (3.4)] and the references therein, or [Lév11] and [Far08, Chapter12] for a case-by-case computation. More precisely, denoting  $E_{ij}$  the elementary matrices in  $M(n, k)$  with  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , the Casimir elements with respect to the scalar product (1.6) write as

$$\begin{aligned} C_{\mathfrak{so}(n)} &= \sum_{1 \leq i < j \leq n} \left( \frac{E_{ij} - E_{ji}}{\sqrt{n}} \right)^{\otimes 2} \\ C_{\mathfrak{su}(n)} &= \frac{1}{n} \sum_{i=1}^n iE_{ii} \otimes iE_{ii} - \frac{1}{n^2} \sum_{i,j=1}^n iE_{ii} \otimes iE_{jj} + \sum_{1 \leq i < j \leq n} \left( \frac{E_{ij} - E_{ji}}{\sqrt{2n}} \right)^{\otimes 2} + \left( \frac{iE_{ij} + iE_{ji}}{\sqrt{2n}} \right)^{\otimes 2} \\ C_{\mathfrak{usp}(n)} &= \frac{1}{2n} \sum_{i=1}^n iE_{ii} \otimes iE_{ii} + jE_{ii} \otimes jE_{ii} + kE_{ii} \otimes kE_{ii} \\ &\quad + \sum_{1 \leq i < j \leq n} \left( \frac{E_{ij} - E_{ji}}{\sqrt{4n}} \right)^{\otimes 2} + \left( \frac{iE_{ij} + iE_{ji}}{\sqrt{4n}} \right)^{\otimes 2} + \left( \frac{jE_{ij} + jE_{ji}}{\sqrt{4n}} \right)^{\otimes 2} + \left( \frac{kE_{ij} + kE_{ji}}{\sqrt{4n}} \right)^{\otimes 2} \end{aligned}$$

see [Lév11, Lemma 1.2] — beware that the tensor product is over  $\mathbb{R}$ , since we deal with real Lie algebras. One can then use the relations of the universal Lie algebra  $U(\mathfrak{k}^{\mathbb{C}})$  to rewrite  $C_{\mathfrak{t}}$  as the sum  $T + N$  of

- an element  $T$  in  $U(\mathfrak{t})$  that acts on  $V^{\lambda}$  by the coefficient of Formula (2.7);
- an element  $N$  that vanishes on the highest weight vector of  $V^{\lambda}$ , because part of the left ideal of  $U(\mathfrak{k}^{\mathbb{C}})$  generated by the elements of  $\mathfrak{n}$ ,  $\mathfrak{n}$  being the nilpotent subalgebra of  $\mathfrak{k}^{\mathbb{C}}$  associated to the choice of the Weyl chamber;

see the argument of [Far08, p. 287] for more details. In the case of a compact symmetric space, the same Formula (2.7) gives the action of  $\Delta^{G/K}$  on  $\mathcal{C}^{\lambda}(G/K)$ . Indeed, remember that the Riemannian structures on  $G$  and  $G/K$  are chosen in such a way that for any  $f \in \mathcal{C}^{\infty}(G)$  that is right  $K$ -invariant,

$$\Delta^{G/K}(f)(gK) = \Delta^G(f)(g).$$

Consider then a function in  $\mathcal{C}^{\lambda}(G/K)$ , viewed as a function on  $G$ . In Definition (2.3),  $f$  appears clearly as a linear combination of matrix coefficients of the spherical representation  $\lambda \in \widehat{G^K}$ , so the previous discussion holds.

**2.3. Densities of a Brownian motion with values in a compact symmetric space.** Let us now see how the previous results can be used to compute the density  $p_t^K(k)$  or  $p_t^X(x)$  of a Brownian motion on a compact Lie group or symmetric space. These densities are in both cases  $K$ -invariant, so they can be written as

$$p_t^K(k) = \sum_{\lambda \in \widehat{K}} a_{\lambda}(t) \widehat{\chi}^{\lambda}(k) \quad \text{or} \quad p_t^X(x) = \sum_{\lambda \in \widehat{G^K}} a_{\lambda}(t) \phi^{\lambda}(x)$$

by using either Peter-Weyl's theorem in the case of conjugacy-invariant functions on  $K$ , or Cartan's theorem in the case of left  $K$ -invariant functions on  $G/K$ . We then apply  $\frac{\Delta}{2} = \frac{dP_t}{dt} \Big|_{t=0}$  to these formulas:

$$\frac{\Delta p_t^K(k)}{2} = \sum_{\lambda \in \widehat{K}} \frac{\kappa_{\lambda}}{2} a_{\lambda}(t) \widehat{\chi}^{\lambda}(k) = \frac{dp_t^K(t)}{dt} = \sum_{\lambda \in \widehat{K}} \frac{da_{\lambda}(t)}{dt} \widehat{\chi}^{\lambda}(k),$$

and similarly in the case of a compact symmetric space. Thus,  $\frac{da_{\lambda}(t)}{dt} = \frac{\kappa_{\lambda}}{2} a_{\lambda}(t)$  and  $a_{\lambda}(t) = a_{\lambda}(0) e^{\frac{\kappa_{\lambda}}{2} t}$  for every class  $\lambda$ . The coefficient  $a_{\lambda}(0)$  is given in the group case by

$$a_{\lambda}(0) = (D^{\lambda})^2 \int_K \widehat{\chi}^{\lambda}(k) \delta_{e_K}(dk) = (D^{\lambda})^2 \widehat{\chi}^{\lambda}(e_K) = (D^{\lambda})^2$$

and in the case of a compact symmetric space of type non-group by

$$a_{\lambda}(0) = D^{\lambda} \left\langle e^{\lambda} \left| \int_G \rho^{\lambda}(g) (e^{\lambda}) \delta_{e_G}(dg) \right. \right\rangle = D^{\lambda} \phi^{\lambda}(e_G) = D^{\lambda}.$$

Combining this with the highest weight theory exposed in the previous paragraph, one obtains

**Proposition 14.** *The density of the law  $\mu_t$  of the Brownian motion traced on a classical simple compact Lie group  $K$  is*

$$p_t^K(k) = \sum_{\lambda \in \hat{K}} e^{-\frac{t}{2} \langle \lambda + 2\rho | \lambda \rangle} \left( \frac{\prod_{\alpha \in \Phi_+} \langle \lambda + \rho | \alpha \rangle}{\prod_{\alpha \in \Phi_+} \langle \rho | \alpha \rangle} \right)^2 \hat{\chi}^\lambda(k),$$

and the density of the Brownian motion traced on a classical simple compact symmetric space  $G/K$  is

$$p_t^X(x) = \sum_{\lambda \in \hat{G}^K} e^{-\frac{t}{2} \langle \lambda + 2\rho | \lambda \rangle} \left( \frac{\prod_{\alpha \in \Phi_+} \langle \lambda + \rho | \alpha \rangle}{\prod_{\alpha \in \Phi_+} \langle \rho | \alpha \rangle} \right) \phi^\lambda(x).$$

Let us now apply this proposition in each classical case (see [BD85], [FH91, Chapter 24] and [Hel78, Chapter 10] for precisions on these computations and on the classical root systems). For convenience, we shall suppose:

- $n \geq 2$  when considering  $SU(n)$ ,  $SU(n)/SO(n)$ ,  $SU(2n)/USp(n)$  or  $SU(n)/S(U(n-q) \times U(q))$ ;
- $n \geq 3$  when considering  $USp(n)$ ,  $USp(n)/U(n)$  or  $USp(n)/(USp(n-q) \times USp(q))$ ;
- $n \geq 10$  when considering<sup>6</sup>  $SO(n)$ ,  $SO(2n)/U(n)$  or  $SO(n)/(SO(n-q) \times SO(q))$ .

This ensures in particular that the root systems and the Schur functions of type B, C and D are not degenerate, and later this will ease certain computations. For Grassmanian varieties, we also suppose by symmetry that  $q \leq \lfloor \frac{n}{2} \rfloor$ .

- special unitary groups  $SU(n, \mathbb{C})$ : a maximal torus is

$$T = \left\{ \text{diag}(z_1, z_2, \dots, z_n) \mid \forall i, z_i \in \mathbb{T} \text{ and } \prod_{i=1}^n z_i = 1 \right\} = \mathbb{T}^n / \mathbb{T} \quad (2.8)$$

and the Weyl group is the symmetric group  $\mathfrak{S}_n$ . The simple roots and the fundamental weights, viewed as elements of  $\mathfrak{t}^*$ , are  $\alpha_i = e^i - e^{i+1}$  and  $\varpi_i = e^1 + \dots + e^i$  for  $i \in \llbracket 1, n-1 \rrbracket$ , where  $e^i$  is the coordinate form on  $\mathfrak{t} = i\mathbb{R}^n$  defined by  $e^i(\text{diag}(it_1, it_2, \dots, it_n)) = t_i$ . Here there is a small subtlety as in  $\mathfrak{t}^*$ ,  $\varpi_n = e^1 + \dots + e^n = 0$ . Therefore, the fundamental weights can also be written as

$$\widetilde{\varpi}_i = \varpi_i - \frac{i}{n} \varpi_n = \frac{n-i}{n} (e^1 + \dots + e^i) - \frac{i}{n} (e^{i+1} + \dots + e^n);$$

this is actually important in order to compute correctly the Casimir coefficients. The dominant weights are then the

$$(\lambda_1 - \lambda_2) \widetilde{\varpi}_1 + \dots + \lambda_{n-1} \widetilde{\varpi}_{n-1} = \lambda_1 e^1 + \dots + \lambda_{n-1} e^{n-1} - |\lambda| \frac{\varpi_n}{n},$$

where  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1})$  is any partition (non-increasing sequence of non-negative integers) of length  $(n-1)$ ; it is then convenient to set  $\lambda_n = 0$ . The half-sum of positive roots is given by  $2\rho = 2(\widetilde{\varpi}_1 + \dots + \widetilde{\varpi}_{n-1}) = \sum_{i=1}^n (n+1-2i)e^i$ , and the scalar product on  $\mathfrak{t}^*$  is  $\frac{1}{n}$  times the usual euclidian scalar product  $\langle e^i | e^j \rangle = \delta_{ij}$ . So,

$$D^\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

<sup>6</sup>For  $SU(2n)/USp(n)$  and  $SO(2n)/U(n)$ , the restriction holds on the “ $2n$ ” parameter of the group of isometries. So for instance, we shall consider the spaces  $SO(2n)/U(n)$  with  $2n \geq 10$ .

and the Casimir coefficient is  $\kappa_\lambda = -\frac{|\lambda|^2}{n^2} + \frac{1}{n} \sum_{i=1}^n \lambda_i^2 + (n+1-2i)\lambda_i$ . The character of the representation of highest weight  $\lambda$  is the Schur function

$$\chi^\lambda(k) = s_\lambda(z_1, \dots, z_n) = \frac{\det(z_i^{\lambda_j + n - j})_{1 \leq i, j \leq n}}{\det(z_i^{n-j})_{1 \leq i, j \leq n}},$$

where  $z_1, \dots, z_n$  are the eigenvalues of  $k$ .

- unitary groups  $U(n, \mathbb{C})$ : though we have chosen to examine only the Brownian motions on simple Lie groups or symmetric spaces, the same work can be performed over the unitary groups, which have a non-trivial and one-dimensional center, and therefore are not semi-simple but reducible. The theory is in fact almost the same as for  $SU(n, \mathbb{C})$ . Irreducible representations of  $U(n, \mathbb{C})$  are labelled by sequences  $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$  in  $\mathbb{Z}^n$ , the action of the torus  $\mathbb{T}^n$  on a corresponding highest weight vector being given by the morphism  $\lambda(z_1, \dots, z_n) = z_1^{\lambda_1} \dots z_n^{\lambda_n}$ . The dimensions and characters are given by the same formulas as before, and the Casimir coefficient is this time  $\frac{1}{n} \sum_{i=1}^n \lambda_i^2 + (n+1-2i)\lambda_i$ .
- spaces of real structures  $SU(n, \mathbb{C})/SO(n, \mathbb{R})$ : the involutive automorphism defining the symmetric pair is  $\theta(g) = \bar{g}$ . The intersection of the torus (2.8) with  $SO(n, \mathbb{R})$  is  $(\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z})$ , the set of diagonal matrices  $\text{diag}(x_1, \dots, x_n)$  with coefficients  $x_i$  in  $\{\pm 1\}$  and determinant  $x_1 \dots x_n = 1$ . The dominant weights  $\lambda = (\lambda_1, \dots, \lambda_n)$  such that  $\lambda(t) = \lambda(x_1, \dots, x_n) = x_1^{\lambda_1} \dots x_n^{\lambda_n} = 1$  for every  $t \in (\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z})$  are those with even parts:

$$\forall i \in \llbracket 1, n \rrbracket, \lambda_i \equiv 0 \pmod{2}.$$

- spaces of quaternionic structures  $SU(2n, \mathbb{C})/USp(n, \mathbb{H})$ :  $\theta(g) = J_{2n} \bar{g} J_{2n}^{-1}$ , where  $J_{2n}$  is the  $(2n \times 2n)$ -skew symmetric matrix

$$J_{2n} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

The intersection of the torus (2.8) (with  $2n$  entries) with  $USp(n, \mathbb{H})$  is the set of diagonal matrices  $\text{diag}(z_1, z_1^{-1}, \dots, z_n, z_n^{-1})$ , with all the  $z_i$ 's in  $\mathbb{T}$ . The dominant weights  $\lambda$  trivial on  $T \cap K$  are then characterized by the condition

$$\forall i \in \llbracket 1, n \rrbracket, \lambda_{2i-1} = \lambda_{2i}.$$

In other words, all the parts are doubled.

- complex Grassmannian varieties  $SU(n, \mathbb{C})/S(U(n-q, \mathbb{C}) \times U(q, \mathbb{C}))$ : this time, it is a little simpler to work with  $U(n, \mathbb{C})/(U(n-q, \mathbb{C}) \times U(q, \mathbb{C}))$ , which is the same space. The involutive automorphism defining the symmetric pair is then  $\theta(g) = K_{n,q} g K_{n,q}$ , where

$$K_{n,q} = \begin{pmatrix} & & T_q \\ & I_{n-2q} & \\ T_q & & \end{pmatrix}$$

and  $T_q$  is the  $(q \times q)$ -anti-diagonal matrix with entries 1 on the anti-diagonal. Notice that this does not correspond to the usual embedding<sup>7</sup> of  $U(n-q, \mathbb{C}) \times U(q, \mathbb{C})$  into  $U(n, \mathbb{C})$ ; but this is the good choice for  $\theta$  so that the usual maximal torus  $T = \mathbb{T}^n$  satisfies the crucial hypothesis “ $T \cap P$  maximal torus in  $P$ ”. The intersection of the torus (2.8) with  $K$  is then the set of diagonal

<sup>7</sup>The usual embedding corresponds to the matrix  $K'_{n,q} = \text{diag}(1, \dots, 1, -1, \dots, -1)$  with  $(n-q)$  entries 1 and  $q$  entries  $-1$ ; it is conjugated to the matrix  $K_{n,q}$  proposed.



matrices  $\text{diag}(z_1, \dots, z_q, z_{q+1}, \dots, z_{n-q}, z_q, \dots, z_1)$  with the  $z_i$ 's in  $\mathbb{T}$ . The dominant weights  $\lambda$  trivial on  $T \cap K$  are then

$$\lambda = (\lambda_1, \dots, \lambda_q, 0, \dots, 0, -\lambda_q, \dots, -\lambda_1).$$

Therefore, they are given by a partition of length  $q$ .

- compact symplectic groups  $\text{USp}(n, \mathbb{H})$ : considering  $\text{USp}(n, \mathbb{H})$  as a subgroup of  $\text{SU}(2n, \mathbb{C})$ , a maximal torus is

$$T = \left\{ \text{diag}(z_1, z_1^{-1}, \dots, z_n, z_n^{-1}) \mid \forall i, z_i \in \mathbb{T} \right\} \quad (2.9)$$

and the Weyl group is the hyperoctahedral group  $\mathfrak{H}_n = (\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n$ , acting by permutation of the  $z_i$ 's and by exchange of  $z_i$  and  $z_i^{-1}$ . The simple roots, viewed as elements of  $\mathfrak{t}^*$ , are  $\alpha_i = e^i - e^{i+1}$  for  $i \in \llbracket 1, n-1 \rrbracket$  and  $\alpha_n = 2e^n$ ; and the fundamental weights are  $\varpi_i = e^1 + \dots + e^i$  for  $i \in \llbracket 1, n \rrbracket$ . Here,  $e^i(\text{diag}(it_1, -it_1, \dots, it_n, -it_n)) = t_i$ . The dominant weights can therefore be written as  $\lambda_1 e^1 + \dots + \lambda_n e^n$ , where  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$  is any partition of length  $n$ . The sum of positive roots is  $2\rho = \sum_{i=1}^n (2n+2-2i)e^i$ , and the scalar product on  $\mathfrak{t}^*$  is  $\frac{1}{2n}$  times the usual euclidian scalar product. Hence,

$$D^\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j},$$

and the Casimir coefficient is  $\kappa_\lambda = \frac{1}{2n} \sum_{i=1}^n \lambda_i^2 + (2n+2-2i)\lambda_i$ . The character of the representation of highest weight  $\lambda$  is the Schur function of type C

$$\chi^\lambda(k) = sc_\lambda(z_1, z_1^{-1}, \dots, z_n, z_n^{-1}) = \frac{\det(z_i^{\lambda_j+n-j+1} - z_i^{-(\lambda_j+n-j+1)})_{1 \leq i, j \leq n}}{\det(z_i^{n-j+1} - z_i^{-(n-j+1)})_{1 \leq i, j \leq n}},$$

where  $z_1^{\pm 1}, \dots, z_n^{\pm 1}$  are the eigenvalues of  $k$  viewed as a matrix in  $\text{SU}(2n, \mathbb{C})$ .

- spaces of complex structures  $\text{USp}(n, \mathbb{H})/\text{U}(n, \mathbb{C})$ : the involutive automorphism associated to the symmetric pair is  $\theta(g) = \bar{g}$  (again, inside  $\text{SU}(2n, \mathbb{C})$ ). This corresponds to the choice  $\mathbb{C} = \mathbb{R} \oplus j\mathbb{R}$  for the inclusion  $\mathbb{C} \subset \mathbb{H}$  and then  $\text{U}(n, \mathbb{C}) \subset \text{USp}(n, \mathbb{H})$ . The intersection of the torus (2.9) with  $\text{U}(n, \mathbb{C})$  is the set of diagonal matrices  $\text{diag}(x_1, x_1, \dots, x_n, x_n)$  with coefficients  $x_i$  in  $\{\pm 1\}$ . The dominant weights  $\lambda = (\lambda_1, \dots, \lambda_n)$  such that  $\lambda(t) = \lambda(x_1, x_1, \dots, x_n, x_n) = x_1^{\lambda_1} \dots x_n^{\lambda_n} = 1$  for every  $t \in (\mathbb{Z}/2\mathbb{Z})^n$  are those with even parts:

$$\forall i \in \llbracket 1, n \rrbracket, \lambda_i \equiv 0 \pmod{2}.$$

- quaternionic Grassmannian varieties  $\text{USp}(n, \mathbb{H})/(\text{USp}(n-q, \mathbb{H}) \times \text{USp}(q, \mathbb{H}))$ :  $\theta(g) = L_{2n,q} g L_{2n,q}$ , where

$$L_{2n,q} = \begin{pmatrix} T_4 & & & \\ & \ddots & & \\ & & T_4 & \\ & & & I_{2n-4q} \end{pmatrix},$$

$T_4$  appearing  $q$  times (with all the computations made inside  $\text{SU}(2n, \mathbb{C})$ ). Again, this peculiar choice is made so that  $T \cap P$  is maximal in  $P$  (of rank  $q$ ), and it does not correspond to the usual embedding of  $\text{USp}(n-q, \mathbb{H}) \times \text{USp}(q, \mathbb{H})$  into  $\text{USp}(n, \mathbb{H})$ . The intersection of  $K$  with the torus (2.9) is the set of diagonal matrices  $\text{diag}(z_1, z_1^{-1}, z_1^{-1}, z_1, \dots, z_q, z_q^{-1}, z_q^{-1}, z_q, z_{2q+1}, z_{2q+1}^{-1}, \dots, z_n, z_n^{-1})$  with the  $z_i$ 's in  $\mathbb{T}$ . The dominant weights  $(\lambda_1, \dots, \lambda_n)$  trivial on  $K \cap T$  write then as

$$\lambda = (\lambda_1, \lambda_1, \dots, \lambda_q, \lambda_q, 0, \dots, 0).$$

Thus, they are given by a partition of length  $q$  (with parts doubled).

- special orthogonal groups  $\mathrm{SO}(n, \mathbb{R})$ : odd and even special orthogonal groups do not have the same kind of root system, and on the other hand,  $\mathrm{SO}(n, \mathbb{R})$  is not simply connected and has for fundamental group  $\mathbb{Z}/2\mathbb{Z}$  for  $n \geq 3$ . So in theory, the arguments previously recalled apply only for the universal cover  $\mathrm{Spin}(n)$ . Nonetheless, most of the results will stay true, and in particular the labeling of the irreducible representations; see the end of [BD85, Chapter 5] for details on this question. In the odd case, a maximal torus in  $\mathrm{SO}(2n+1, \mathbb{R})$  is

$$T = \left\{ \mathrm{diag}(R_{\theta_1}, \dots, R_{\theta_n}, 1) \mid \forall i, R_{\theta_i} = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix} \in \mathrm{SO}(2, \mathbb{R}) \right\} \quad (2.10)$$

and the Weyl group is again the hyperoctahedral group  $\mathfrak{H}_n$ . The simple roots are  $\alpha_i = e^i - e^{i+1}$  for  $i \in \llbracket 1, n-1 \rrbracket$ , and  $\alpha_n = e^n$ ; and the fundamental weights are  $\varpi_i = e^1 + \dots + e^i$  for  $i \in \llbracket 1, n-1 \rrbracket$ , and  $\varpi_n = \frac{1}{2}(e^1 + \dots + e^n)$ . Here,

$$e^i \left( \mathrm{diag} \left( \begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -a_n \\ a_n & 0 \end{pmatrix}, 0 \right) \right) = a_i$$

and it corresponds to the morphism  $\mathrm{diag}(R_{\theta_1}, \dots, R_{\theta_n}, 1) \mapsto e^{i\theta_i}$ . The dominant weights are then  $\lambda_1 e^1 + \dots + \lambda_n e^n$ , where  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$  is either a partition of length  $n$ , or an half-partition of length  $n$ , where by half-partition we mean a non-increasing sequence of half-integers in  $\mathbb{N}' = \mathbb{N} + 1/2$ . The sum of positive roots is  $2\rho = \sum_{i=1}^n (2n+1-2i)e^i$ , and the scalar product on  $\mathfrak{t}^*$  is  $\frac{1}{(2n+1)}$  times the usual euclidian scalar product  $\langle e^i \mid e^j \rangle = \delta_{ij}$ . So,

$$D^\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 1 - i - j}{2n + 1 - i - j},$$

and the Casimir coefficient is  $\kappa_\lambda = \frac{1}{2n+1} \sum_{i=1}^n \lambda_i^2 + (2n+1-2i)\lambda_i$ . The character of the representation of highest weight  $\lambda$  is the Schur function of type B

$$\chi^\lambda(k) = sb_\lambda(z_1, z_1^{-1}, \dots, z_n, z_n^{-1}, 1) = \frac{\det(z_i^{\lambda_j + n - j + 1/2} - z_i^{-(\lambda_j + n - j + 1/2)})_{1 \leq i, j \leq n}}{\det(z_i^{n - j + 1/2} - z_i^{-(n - j + 1/2)})_{1 \leq i, j \leq n}},$$

where  $z_1^{\pm 1}, \dots, z_n^{\pm 1}, 1$  are the eigenvalues of  $k$ . In the even case, a maximal torus in  $\mathrm{SO}(2n, \mathbb{R})$  is

$$T = \left\{ \mathrm{diag}(R_{\theta_1}, \dots, R_{\theta_n}) \mid \forall i, R_{\theta_i} = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix} \in \mathrm{SO}(2, \mathbb{R}) \right\} \quad (2.11)$$

and the Weyl group is  $\mathfrak{H}_n^+$ , the subgroup of  $\mathfrak{H}_n$  of index 2 consisting in signed permutations with an even number of signs  $-1$ . The simple roots are  $\alpha_i = e^i - e^{i+1}$  for  $i \in \llbracket 1, n-1 \rrbracket$  and  $\alpha_n = e^{n-1} + e^n$ ; and the fundamental weights are  $\varpi_i = e^1 + \dots + e^i$  for  $i \in \llbracket 1, n-2 \rrbracket$  and  $\varpi_{n-1, n} = \frac{1}{2}(e^1 + \dots + e^{n-1} \pm e^n)$ . The dominant weights are then  $\lambda_1 e^1 + \dots + \lambda_{n-1} e^{n-1} + \varepsilon \lambda_n e^n$ , where  $\varepsilon$  is a sign and  $(\lambda_1 \geq \dots \geq \lambda_n)$  is either a partition or an half-partition of length  $n$ . The sum of positive roots is  $2\rho = \sum_{i=1}^n (2n-2i)e^i$ , so

$$D^\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \frac{\lambda_i + \lambda_j + 2n - i - j}{2n - i - j},$$

and  $\kappa_\lambda = \frac{1}{2n} \sum_{i=1}^n \lambda_i^2 + (2n-2i)\lambda_i$ . The irreducible characters are the Schur functions of type C

$$\begin{aligned} \chi^\lambda(k) &= sd_\lambda(z_1, z_1^{-1}, \dots, z_n, z_n^{-1}) \\ &= \frac{\det(z_i^{\lambda_j + n - j} - z_i^{-(\lambda_j + n - j)})_{1 \leq i, j \leq n} + \det(z_i^{\lambda_j + n - j} + z_i^{-(\lambda_j + n - j)})_{1 \leq i, j \leq n}}{\det(z_i^{n - j} + z_i^{-(n - j)})_{1 \leq i, j \leq n}}, \end{aligned}$$

where  $z_1^{\pm 1}, \dots, z_n^{\pm 1}$  are the eigenvalues of  $k$ .

- real Grassmannian varieties  $\mathrm{SO}(n, \mathbb{R})/(\mathrm{SO}(n-q, \mathbb{R}) \times \mathrm{SO}(q, \mathbb{R}))$ : in this case and for the next one, one cannot directly apply Theorem 13, since  $\mathrm{SO}(n, \mathbb{R})$  is not simply connected. A rigorous way to deal with this problem is to first look at quotients of the spin group  $\mathrm{Spin}(n)$ ; of course

this will not be very practical, as  $\text{Spin}(n)$  has no “natural” matrix representation. Nonetheless, consider the Grassmannian variety of *non-oriented* vector spaces

$$\text{Gr}^\pm(n, q, \mathbb{R}) \simeq \text{Spin}(n)/(\text{Spin}(n-q) \times \text{Spin}(q));$$

$\text{Gr}(n, q, \mathbb{R})$  is a 2-fold covering of  $\text{Gr}^\pm(n, q, \mathbb{R})$ . The defining involution of  $\text{Gr}^\pm(n, q, \mathbb{R})$  corresponds to the involution of  $\text{SO}(n, \mathbb{R})$  given by  $\theta(g) = N_{n,q} g N_{n,q}$ , where

$$N_{n,q} = \begin{pmatrix} T_2 & & & \\ & \ddots & & \\ & & T_2 & \\ & & & I_{n-2q} \end{pmatrix}$$

with  $q$  blocks  $T_2$ . With  $T$  as in Equations (2.10) and (2.11), the intersection  $K \cap T$  is then  $(\mathbb{Z}/2\mathbb{Z})^q \times (\text{SO}(2, \mathbb{R}))^{\lfloor \frac{n}{2} \rfloor - q}$ , so the dominant weights trivial on  $K \cap T$  write as

$$\lambda = (\lambda_1, \dots, \lambda_q, 0, \dots, 0) \quad ; \quad \forall i \in \llbracket 1, q \rrbracket, \quad \lambda_i \equiv 0 \pmod{2}.$$

They are therefore given by an integer partition of length  $q$ , with all parts even. Now, for the simply connected Grassmannian variety  $\text{Gr}(n, q, \mathbb{R})$ , there are twice as many spherical representations, as  $K \cap T$  is in this case isomorphic to  $((\mathbb{Z}/2\mathbb{Z})^q/(\mathbb{Z}/2\mathbb{Z})) \times \mathbb{T}^{\lfloor \frac{n}{2} \rfloor - q}$ , instead of  $(\mathbb{Z}/2\mathbb{Z})^q \times \mathbb{T}^{\lfloor \frac{n}{2} \rfloor - q}$ . Therefore, the condition of parity is now

$$\forall i, j \in \llbracket 1, q \rrbracket, \quad \lambda_i \equiv \lambda_j \pmod{2}.$$

Thus, spherical weights correspond to partitions of length  $q$  with all parts odd, or all parts even.

- spaces of complex structures  $\text{SO}(2n, \mathbb{R})/\text{U}(n, \mathbb{C})$ : denoting  $U_4 = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$ , a convenient choice for  $\theta$  and therefore the embedding  $\text{U}(n, \mathbb{C}) \hookrightarrow \text{SO}(2n, \mathbb{R})$  is  $\theta(g) = M_n g M_n^{-1}$ , where

$$M_n = \begin{pmatrix} U_4 & & \\ & \ddots & \\ & & U_4 \end{pmatrix} \text{ if } n \text{ is even} \quad ; \quad M_n = \begin{pmatrix} U_4 & & \\ & \ddots & \\ & & U_4 \\ & & & I_2 \end{pmatrix} \text{ if } n \text{ is odd}$$

with each time  $m = \lfloor \frac{n}{2} \rfloor$  blocks  $U_4$ . We skip here the technicalities related to the non simple-connectedness of  $\text{SO}(2n, \mathbb{R})$ ; again they can be solved by considering the space as a quotient of  $\text{Spin}(2n)$ . In the end, the dominant weights  $\lambda$  trivial on the intersection  $K \cap T$  are given by

$$\lambda = (\lambda_1, \lambda_1, \dots, \lambda_m, \lambda_m) \text{ or } \lambda = (\lambda_1, \lambda_1, \dots, \lambda_m, \lambda_m, 0)$$

that is to say a partition with all non-zero parts that are doubled.

*Remark.* In each case, the “degree of freedom” in the choice of partitions labeling the irreducible or spherical representations is exactly the rank of the Riemannian variety under consideration, that is to say the maximal dimension of flat totally geodesic sub-manifolds.

*Remark.* The special embeddings detailed in this paragraph were only useful to find out which highest weights were corresponding to spherical representations (*via* Cartan-Helgason’s Theorem 13, which requires  $K$  to be chosen so that  $P \cap T$  is maximal in  $T$ ). However, from now on, one can come back to more traditional embeddings, in particular in the cumbersome case of Grassmannian varieties.

In the following, we denote:  $\mathfrak{P}_n$  the set of partitions of length  $n$ ;  $\mathfrak{Z}_n$  the set of non-decreasing sequences of (possibly negative) integers;  $\frac{1}{2}\mathfrak{P}_n$  the set of partitions and half-partitions of length  $n$ ;  $2\mathfrak{P}_n$  the set of partitions of length  $n$  with even parts;  $2\mathfrak{P}_n \boxplus 1$  the set of partitions of length  $n$  with odd parts; and  $\mathfrak{P}\mathfrak{P}_n$  the set of partitions of length  $n$  and with all non-zero parts doubled. It is understood that if  $i$  is too big, then  $\lambda_i = 0$  for a partition or an half-partition  $\lambda$  of prescribed length.

**Theorem 15.** *The density of the law  $\mu_t$  of the Brownian motion traced on a classical simple compact Lie group writes explicitly as follows:*

$$\begin{aligned}
& \sum_{\lambda \in \mathfrak{Y}_{n-1}} e^{-\frac{t}{2n} \left( -\frac{|\lambda|^2}{n} + \sum_{i=1}^{n-1} \lambda_i^2 + (n+1-2i)\lambda_i \right)} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right) s_\lambda(k); \\
& \sum_{\lambda \in \mathfrak{Z}_n} e^{-\frac{t}{2n} \sum_{i=1}^n \lambda_i^2 + (n+1-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right) s_\lambda(k); \\
& \sum_{\lambda \in \mathfrak{Y}_n} e^{-\frac{t}{4n} \sum_{i=1}^n \lambda_i^2 + (2n+2-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j} \right) sc_\lambda(k); \\
& \sum_{\lambda \in \frac{1}{2}\mathfrak{Y}_n} e^{-\frac{t}{4n+2} \sum_{i=1}^n \lambda_i^2 + (2n+1-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 1 - i - j}{2n + 1 - i - j} \right) sb_\lambda(k); \\
& \sum_{\lambda \in \frac{1}{2}\mathfrak{Y}_n} e^{-\frac{t}{4n} \sum_{i=1}^n \lambda_i^2 + (2n-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n - i - j)}{(j - i)(2n - i - j)} \right) (sd_\lambda(k) + sd_{\varepsilon\lambda}(k))
\end{aligned}$$

respectively for special unitary groups  $SU(n, \mathbb{C})$ , unitary groups  $U(n, \mathbb{C})$ , symplectic groups  $USp(n, \mathbb{H})$ , odd special orthogonal groups  $SO(2n+1, \mathbb{R})$ , and even special orthogonal groups  $SO(2n, \mathbb{R})$ . In this last case,  $\varepsilon\lambda = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$ , and it is agreed that  $sd_\lambda + sd_{\varepsilon\lambda}$  stands for  $sd_\lambda$  if  $\lambda_n = 0$  and therefore  $\varepsilon\lambda = \lambda$ .

We denote generically  $\phi_\lambda(x)$  a zonal spherical function associated to a spherical representation (the function depends of course of the implicit type of the space considered). The density of the law  $\mu_t$  of the Brownian motion traced on a classical simple compact symmetric space writes then as follows:

$$\begin{aligned}
& \sum_{\lambda \in 2\mathfrak{Y}_{n-1}} e^{-\frac{t}{2n} \left( -\frac{|\lambda|^2}{n} + \sum_{i=1}^{n-1} \lambda_i^2 + (n+1-2i)\lambda_i \right)} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \phi_\lambda(x); \\
& \sum_{\lambda \in \mathfrak{Y}_{2n-1}} e^{-\frac{t}{4n} \left( -\frac{|\lambda|^2}{2n} + \sum_{i=1}^{2n-2} \lambda_i^2 + (2n+1-2i)\lambda_i \right)} \left( \prod_{1 \leq i < j \leq 2n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \phi_\lambda(x); \\
& \sum_{\lambda \in \mathfrak{Y}_q} e^{-\frac{t}{n} \sum_{i=1}^q \lambda_i^2 + (n+1-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \phi_\lambda(x); \\
& \sum_{\lambda \in 2\mathfrak{Y}_n} e^{-\frac{t}{4n} \sum_{i=1}^n \lambda_i^2 + (2n+2-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j} \right) \phi_\lambda(x); \\
& \sum_{\lambda \in \mathfrak{Y}_{2q}} e^{-\frac{t}{4n} \sum_{i=1}^{2q} \lambda_i^2 + (2n+2-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j} \right) \phi_\lambda(x); \\
& \sum_{\lambda \in \mathfrak{Y}_n} e^{-\frac{t}{4n} \sum_{i=1}^n \lambda_i^2 + (2n-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n - i - j)}{(j - i)(2n - i - j)} \right) \phi_\lambda(x);
\end{aligned}$$

$$\sum_{\lambda \in 2\mathfrak{Y}_q \sqcup 2\mathfrak{Y}_q \boxplus 1} e^{-\frac{t}{4n+2} \sum_{i=1}^q \lambda_i^2 + (2n+1-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 1 - i - j}{2n + 1 - i - j} \right) \phi_\lambda(x);$$

$$\sum_{\lambda \in 2\mathfrak{Y}_q \sqcup 2\mathfrak{Y}_q \boxplus 1} e^{-\frac{t}{4n} \sum_{i=1}^q \lambda_i^2 + (2n-2i)\lambda_i} \left( \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n - i - j)}{(j - i)(2n - i - j)} \right) \phi_\lambda(x)$$

for real structures  $\mathrm{SU}(n, \mathbb{C})/\mathrm{SO}(n, \mathbb{R})$ , quaternionic structures  $\mathrm{SU}(2n, \mathbb{C})/\mathrm{USp}(n, \mathbb{H})$ , complex Grassmannian varieties  $\mathrm{Gr}(n, q, \mathbb{C})$ , complex structures  $\mathrm{USp}(n, \mathbb{H})/\mathrm{U}(n, \mathbb{C})$ , quaternionic Grassmannian varieties  $\mathrm{Gr}(n, q, \mathbb{H})$ , complex structures  $\mathrm{SO}(2n, \mathbb{R})/\mathrm{U}(n, \mathbb{C})$ , odd real Grassmannian varieties  $\mathrm{Gr}(2n+1, q, \mathbb{R})$  and even real Grassmannian varieties  $\mathrm{Gr}(2n, q, \mathbb{R})$ .

*Remark.* In the case of complex Grassmannian varieties, it is understood that  $\lambda_{n+1-i} = -\lambda_i$  as explained before. We have not tried to reduce the expressions in the previous formulas, so some simplifications can be made by replacing the indexing sets of type  $2\mathfrak{Y}_p$  or  $\mathfrak{Y}\mathfrak{Y}_p$  by  $\mathfrak{Y}_p$ .

*Example* (Brownian motions on spheres and projective spaces). Let us examine the case  $q = 1$  for Grassmannian varieties: it corresponds to real spheres  $\mathbb{S}^n(\mathbb{R}) = \mathrm{SO}(n+1, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$ , to complex projective spaces  $\mathbb{P}^n(\mathbb{C}) = \mathrm{SU}(n+1, \mathbb{C})/(\mathrm{S}(\mathrm{U}(n, \mathbb{C}) \times \mathrm{U}(1, \mathbb{C})))$  and to quaternionic projective spaces  $\mathbb{P}^n(\mathbb{H}) = \mathrm{USp}(n+1, \mathbb{H})/(\mathrm{USp}(n, \mathbb{H}) \times \mathrm{USp}(1, \mathbb{H}))$ . In each case, spherical representations are labelled by a single integer  $k \in \mathbb{N}$ . So, the densities are:

$$p_t^{\mathbb{S}^n(\mathbb{R})}(x) = \sum_{k=0}^{\infty} e^{-\frac{k(k+n-1)t}{2n+2}} \frac{(n-2+k)!}{(n-1)!k!} (2k+n-1) \phi_{n,k}^{\mathbb{R}}(x); \quad (2.12)$$

$$p_t^{\mathbb{P}^n(\mathbb{C})}(x) = \sum_{k=0}^{\infty} e^{-\frac{k(k+n)t}{n+1}} \frac{((n-1+k)!)^2}{(n-1)!n!(k!)^2} (2k+n) \phi_{n,k}^{\mathbb{C}}(x); \quad (2.13)$$

$$p_t^{\mathbb{P}^n(\mathbb{H})}(x) = \sum_{k=0}^{\infty} e^{-\frac{k(k+2n+1)t}{2(n+1)}} \frac{(2n+k)!(2n-1+k)!}{(2n+1)!(2n-1)!(k+1)!k!} (2k+2n+1) \phi_{n,k}^{\mathbb{H}}(x). \quad (2.14)$$

In particular, one recovers the well-known fact that, up to the aforementioned normalization factor  $(n+1)$ , the eigenvalues of the Laplacian on the  $n$ -sphere are the  $k(k+n-1)$ , each with multiplicity  $\frac{(n-2+k)!}{(n-1)!k!} (2k+n-1)$ ; see *e.g.* [SC94, §3.3].

*Example* (Torus and Fourier analysis). Take the circle  $\mathbb{T} = \mathrm{U}(1, \mathbb{C}) = \mathbb{S}^1(\mathbb{R})$ . The Brownian motion on  $\mathbb{T}$  is the projection of the real Brownian motion of density  $p_t^{\mathbb{R}}(\theta) = \frac{1}{\sqrt{2\pi t}} e^{-\theta^2/2t}$  by the map  $\theta \mapsto e^{i\theta}$ . Thus,

$$p_t^{\mathbb{T}}(e^{i\theta}) = 2\pi \sum_{m=-\infty}^{\infty} p_t^{\mathbb{R}}(\theta + 2m\pi) = \sqrt{\frac{2\pi}{t}} \sum_{m=-\infty}^{\infty} e^{-\frac{(\theta+2m\pi)^2}{2t}} = \sqrt{\frac{2\pi}{t}} S(\theta, t).$$

The series  $S(\theta, t)$  is smooth and  $2\pi$ -periodic, so it is equal to its Fourier series  $\sum_{n=-\infty}^{\infty} c_k(S(t)) e^{ki\theta}$ , with

$$c_k(S(t)) = \sum_{m=-\infty}^{\infty} \int_0^{2\pi} e^{-\frac{(\theta+2m\pi)^2}{2t}} e^{-ki\theta} \frac{d\theta}{2\pi} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{y^2}{2t} - kiy} dy = \sqrt{\frac{t}{2\pi}} e^{-\frac{k^2 t}{2}}.$$

Thus, the density of the Brownian motion on the circle with respect to the Haar measure  $\frac{d\theta}{2\pi}$  is

$$p_t^{\mathbb{T}}(e^{i\theta}) = \sum_{k=-\infty}^{\infty} e^{-\frac{k^2 t}{2}} e^{ki\theta} = 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{k^2 t}{2}} \cos k\theta,$$

Since  $s_{(k)}(e^{i\theta}) = e^{ki\theta}$ , this is indeed a specialization of the second formula of Theorem 15, for  $\mathrm{U}(1, \mathbb{C})$ .

*Example* (Brownian motion on the 3-dimensional sphere). Consider the Brownian motion on  $\mathrm{USp}(1, \mathbb{H})$ , which is also  $\mathrm{SU}(2, \mathbb{C})$  by one of the exceptional isomorphisms. The specialization of the first formula of Theorem 15 for  $\mathrm{SU}(2, \mathbb{C})$  gives

$$p_t^{\mathrm{SU}(2, \mathbb{C})}(g) = \sum_{k=0}^{\infty} e^{-\frac{k(k+2)}{8}t} (k+1) \frac{\sin(k+1)\theta}{\sin \theta},$$

if  $e^{\pm i\theta}$  are the eigenvalues of  $g \in \mathrm{SU}(2, \mathbb{C})$ . It agrees with the example<sup>8</sup> of [Lia04a, §4], and also with Formula (2.12) when  $n = 3$ , since the group of unit quaternions is topologically a 3-sphere.

*Remark.* The previous examples show that the restrictions  $n \geq n_0$  are not entirely necessary for the formulas of Theorem 15 to hold. One should only beware that the root systems of type  $B_1$ ,  $C_1$ ,  $D_1$  and  $D_2$  are somewhat degenerated, and that the dominant weights do not have the same indexing set as for  $B_{n \geq 2}$  or  $C_{n \geq 2}$  or  $D_{n \geq 3}$ . For instance, for the special orthogonal group  $\mathrm{SO}(3, \mathbb{R})$ , the only positive root is  $e^1$ , and the only fundamental weight is also  $e^1$ . Consequently, irreducible representations have highest weights  $k e^1$  with  $k \in \mathbb{N}$ ; the dimension of the representation of label  $k$  is  $2k + 1$ , and the corresponding character is again  $\frac{\sin(k+1)\theta}{\sin \theta}$  if  $e^{i\theta}$  and  $e^{-i\theta}$  are the non-trivial eigenvalues of the considered rotation. So

$$p_t^{\mathrm{SO}(3, \mathbb{R})}(g) = \sum_{k=0}^{\infty} e^{-\frac{k(k+1)}{3}t} (2k+1) \frac{\sin(k+1)\theta}{\sin \theta}$$

if  $g$  is a rotation of angle  $\theta$  around some axis.

---

<sup>8</sup>Notice that the space is dilated in Liao's paper by  $2\pi$ , whence the factor  $(2\pi)^2 = 4\pi^2$  between our time exponents and the ones given by Liao.

## 3. UPPER BOUNDS AFTER THE CUT-OFF TIME

Let  $\mu$  be a probability measure on a compact Lie group  $K$  or compact symmetric space  $G/K$ , that is absolutely continuous with respect to the Haar measure  $\eta$ , and with density  $p$ . Cauchy-Schwarz inequality ensures that

$$4 d_{\text{TV}}(\mu, \eta)^2 = \left( \int_X |p(x) - 1| dx \right)^2 \leq \int_X |p(x) - 1|^2 dx = \|p - 1\|_{\mathcal{L}^2(X)}^2.$$

The discussion of Section 2 allows now to relate the right-hand side of this inequality with the harmonic analysis on  $X$ . Let us first treat the case of a compact Lie group  $K$ . If one supposes that  $p$  is invariant by conjugacy, then Parseval's identity (2.2) shows that the right-hand side is  $\sum_{\lambda \in \widehat{K}} |\chi^\lambda(p - 1)|^2$ . However, by orthogonality of characters, for any non-trivial irreducible representation of  $K$  — *i.e.*, not equal to  $\mathbf{1}_K : k \in K \mapsto 1$  — one has

$$\chi^\lambda(1) = \int_K \chi^\lambda(k) dk = \int_K \chi^\lambda(k) \chi^{1_K}(k^{-1}) dk = 0.$$

On the other hand, for any measure  $\mu$  on the group,  $\chi^{1_K}(\mu) = \int_K \chi^{1_K}(k) \mu(dk) = \int_K \mu(dk) = 1$ . Hence, the inequality now takes the form

$$4 d_{\text{TV}}(\mu, \eta_K)^2 \leq \sum'_{\lambda \in \widehat{K}} |\chi^\lambda(p)|^2,$$

where the  $'$  indicates that we remove the trivial representation from the summation. Similarly, on a compact symmetric space  $G/K$ , supposing that  $p$  is  $K$ -invariant, Parseval's identity reads  $\|p - 1\|_{\mathcal{L}^2(G/K)}^2 = \sum_{\lambda \in \widehat{G^K}} D^\lambda |\phi^\lambda(p - 1)|^2$ . However, for any non-trivial representation  $\lambda$ ,

$$\phi^\lambda(1) = \left\langle e^\lambda \left| \int_G \rho^\lambda(g)(e^\lambda) dg \right. \right\rangle = 0.$$

Indeed, using only elementary properties of the Haar measure, one sees that  $\widehat{1}(\lambda) = \int_G \rho^\lambda(g) dg = 0$ , because it is a projector and it has trace  $\chi^\lambda(1) = 0$ . So again, the previous inequality can be simplified and it becomes

$$4 d_{\text{TV}}(\mu, \eta_{G/K})^2 \leq \sum'_{\lambda \in \widehat{G^K}} D^\lambda |\phi^\lambda(p)|^2.$$

In the setting and with the notations of Proposition 14, a bound at time  $t$  on  $4 d_{\text{TV}}(\mu_t, \eta_K)^2$  (respectively, on  $4 d_{\text{TV}}(\mu_t, \eta_{G/K})^2$ ) is then

$$\sum'_{\lambda \in \widehat{K}} e^{-t\langle \lambda + 2\rho | \lambda \rangle} (D^\lambda)^2 \quad ; \quad \text{respectively,} \quad \sum'_{\lambda \in \widehat{G^K}} e^{-t\langle \lambda + 2\rho | \lambda \rangle} D^\lambda.$$

**Proposition 16.** *In every classical case,  $4 d_{\text{TV}}(\mu_t, \text{Haar})^2$  is bounded by  $\sum'_{\lambda \in W_n} A_n(\lambda) e^{-t B_n(\lambda)}$ , where the indexing sets  $W_n$  and the constants  $A_n(\lambda)$  and  $B_n(\lambda)$  are recalled in the table on page 30.*

*Remark.* One can shorten all these formulas by introducing the modified weight coordinates

- $\mu_i = \lambda_i + \frac{n+1-2i}{2}$  in type  $A_{n-1}$ , *i.e.*, for  $\text{SU}(n, \mathbb{C})$ ;
- $\mu_i = \lambda_i + n + \frac{1}{2} - i$  in type  $B_n$ , *i.e.*, for  $\text{SO}(2n+1, \mathbb{R})$ ;
- $\mu_i = \lambda_i + n + 1 - i$  in type  $C_n$ , *i.e.*, for  $\text{USp}(n, \mathbb{H})$ ;
- $\mu_i = \lambda_i + n - i$  in type  $D_n$ , *i.e.*, for  $\text{SO}(2n, \mathbb{R})$ .



$K$ or $G/K$	$W_n$	$A_n(\lambda)$	$B_n(\lambda)$
$\mathrm{SO}(2n+1, \mathbb{R})$	$\frac{1}{2}\mathfrak{Y}_n$	$\left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 1 - i - j}{2n + 1 - i - j} \right)^2$	$\frac{1}{2n+1} \left( \sum_{i=1}^n \lambda_i^2 + (2n+1-2i)\lambda_i \right)$
$\mathrm{SO}(2n, \mathbb{R})$	$\frac{1}{2}\mathfrak{Y}_n$	$2 \left( \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n - i - j)}{(j - i)(2n - i - j)} \right)^2$	$\frac{1}{2n} \left( \sum_{i=1}^n \lambda_i^2 + (2n-2i)\lambda_i \right)$
$\mathrm{SU}(n, \mathbb{C})$	$\mathfrak{Y}_{n-1}$	$\left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right)^2$	$\frac{1}{n} \left( -\frac{ \lambda ^2}{n} + \sum_{i=1}^{n-1} \lambda_i^2 + (n+1-2i)\lambda_i \right)$
$\mathrm{USp}(n, \mathbb{H})$	$\mathfrak{Y}_n$	$\left( \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j} \right)^2$	$\frac{1}{2n} \left( \sum_{i=1}^n \lambda_i^2 + (2n+2-2i)\lambda_i \right)$
$\mathrm{Gr}(2n+1, q, \mathbb{R})$	$2\mathfrak{Y}_q \sqcup 2\mathfrak{Y}_q \sqcup 1$	$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 1 - i - j}{2n + 1 - i - j}$	$\frac{1}{2n+1} \left( \sum_{i=1}^q \lambda_i^2 + (2n+1-2i)\lambda_i \right)$
$\mathrm{Gr}(2n, q, \mathbb{R})$	$2\mathfrak{Y}_q \sqcup 2\mathfrak{Y}_q \sqcup 1$	$\prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n - i - j)}{(j - i)(2n - i - j)}$	$\frac{1}{2n} \left( \sum_{i=1}^q \lambda_i^2 + (2n-2i)\lambda_i \right)$
$\mathrm{Gr}(n, q, \mathbb{C})$	$\mathfrak{Y}_q$	$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$	$\frac{2}{n} \left( \sum_{i=1}^q \lambda_i^2 + (n+1-2i)\lambda_i \right)$
$\mathrm{Gr}(n, q, \mathbb{H})$	$\mathfrak{Y}_q \mathfrak{Y}_{2q}$	$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j}$	$\frac{1}{2n} \left( \sum_{i=1}^{2q} \lambda_i^2 + (2n+2-2i)\lambda_i \right)$
$\mathrm{SO}(2n, \mathbb{R})/\mathrm{U}(n, \mathbb{C})$	$\mathfrak{Y}_n \mathfrak{Y}_n$	$\prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n - i - j)}{(j - i)(2n - i - j)}$	$\frac{1}{2n} \left( \sum_{i=1}^n \lambda_i^2 + (2n-2i)\lambda_i \right)$
$\mathrm{SU}(n, \mathbb{C})/\mathrm{SO}(n, \mathbb{R})$	$2\mathfrak{Y}_{n-1}$	$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$	$\frac{1}{n} \left( -\frac{ \lambda ^2}{n} + \sum_{i=1}^{n-1} \lambda_i^2 + (n+1-2i)\lambda_i \right)$
$\mathrm{SU}(2n, \mathbb{C})/\mathrm{USp}(n, \mathbb{H})$	$\mathfrak{Y}_n \mathfrak{Y}_{2n-1}$	$\prod_{1 \leq i < j \leq 2n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$	$\frac{1}{2n} \left( -\frac{ \lambda ^2}{2n} + \sum_{i=1}^{2n-2} \lambda_i^2 + (2n+1-2i)\lambda_i \right)$
$\mathrm{USp}(n, \mathbb{H})/\mathrm{U}(n, \mathbb{C})$	$2\mathfrak{Y}_n$	$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j}$	$\frac{1}{2n} \left( \sum_{i=1}^n \lambda_i^2 + (2n+2-2i)\lambda_i \right)$

Then,  $D^\lambda$  is equal to

$$\begin{aligned} & \frac{V(\mu_1, \mu_2, \dots, \mu_n)}{V(1, 2, \dots, n)} && \text{in type } A_{n-1}; \\ & \left( \frac{2^{n^2} \prod_{i=1}^n \mu_i}{(2n-1)!!} \right) \frac{V(\mu_1^2, \dots, \mu_n^2)}{V((2n-1)^2, \dots, 1^2)} && \text{in type } B_n; \\ & \left( \frac{\prod_{i=1}^n \mu_i}{n!} \right) \frac{V(\mu_1^2, \dots, \mu_n^2)}{V(n^2, \dots, 1^2)} && \text{in type } C_n; \\ & \frac{V(\mu_1^2, \dots, \mu_n^2)}{V((n-1)^2, \dots, 0^2)} && \text{in type } D_n; \end{aligned}$$

where  $V(x_1, \dots, x_n)$  denotes the Vandermonde determinant. Also,  $B_n(\lambda)$  is always an affine function (depending on  $n$ ) of  $\sum_{i=1}^n \mu_i^2$  in type B, C and D; and of  $\sum_{i=1}^n \mu_i^2 - \frac{1}{n} (\sum_{i=1}^n \mu_i)^2$  in type A.

This section is now organized as follows. In §3.1, we compute the weights that minimize  $B_n(\lambda)$ ; they will give the correct order of decay of the whole series after cut-off time. In §3.2, we then show case-by-case that all the other terms of the series  $S_n(t)$  of Proposition 16 can be controlled uniformly. Essentially, we adapt the arguments of [Ros94, Por96a, Por96b], though we also introduce new computational tricks. As explained in the introduction, the main reason why one has a good control over  $S_n(t)$  after cut-off time is that each term  $T_n(\lambda, t) = A_n(\lambda) e^{-t B_n(\lambda)}$  of the series  $S_n(t)$  stays bounded when  $t = t_{\text{cut-off}}$ , for every  $n$ , every class  $\lambda$  and in every case. We have unfortunately not found a way to factorize all the computations needed to prove this, so each case will have to be treated separately. However, the scheme of the proof will always be the same, and the reader will find the main arguments in §3.2.1 (for symplectic groups and their quotients), so he can safely skip §3.2.2-3.2.4 if he does not want to see the minor modifications needed to treat the other cases.

*Remark.* The bounds obtained on the series  $S_n(t)$  in §3.2 are not at all optimal, and in particular one can conjecture that the exponent of  $n$  in these bounds can at least be multiplied by a factor 2. A possible way to improve our bounds would be to use the following alternative approach to the control of the series  $S_n(t)$ ; it is related to the problem of minimization of the *logarithmic potential*

$$\iint_{x \neq y} \log \frac{1}{|x - y|} \nu(dx) \nu(dy) + \int_{\mathbb{R}} P(x) \nu(dx)$$

of a probability measure  $\nu$  on the real line (cf. [ST97]). Set  $t = (1 + \varepsilon) t_{\text{cut-off}}$ , and consider only the case of compact groups. The crucial remark is that up to some explicit constant  $f_n(t)$ , the general term  $T_n(\lambda, t)$  of the series  $S_n(t)$  is equal to the exponential of a logarithmic potential of either the discrete measure  $\sum_{i=1}^n \delta_{\mu_i}$  in the case of unitary groups, or of the discrete measure  $\sum_{i=1}^n \delta_{(\mu_i)^2}$  in the case of special orthogonal groups or compact symplectic groups (the  $\mu'_i$ s are the modified weight coordinates):

$$\begin{aligned} T_n(\lambda, t) &\propto V(\mu_1, \dots, \mu_n)^2 e^{-\frac{t}{n} \sum_{i=1}^n \mu_i^2} && \text{when } K = \text{U}(n); \\ T_n(\lambda, t) &\propto \left( \prod_{i=1}^n \mu_i \right)^2 V(\mu_1^2, \dots, \mu_n^2)^2 e^{-\frac{t}{2n+1} \sum_{i=1}^n \mu_i^2} && \text{when } K = \text{SO}(2n+1); \\ T_n(\lambda, t) &\propto \left( \prod_{i=1}^n \mu_i \right)^2 V(\mu_1^2, \dots, \mu_n^2)^2 e^{-\frac{t}{2n} \sum_{i=1}^n \mu_i^2} && \text{when } K = \text{USp}(n); \\ T_n(\lambda, t) &\propto V(\mu_1^2, \dots, \mu_n^2)^2 e^{-\frac{t}{2n} \sum_{i=1}^n \mu_i^2} && \text{when } K = \text{SO}(2n). \end{aligned}$$

Suppose for a moment that the  $\mu_i$ 's are arbitrary (ordered) *real* numbers, instead of integers or half-integers. Then one can use the following well-known results on discrete minimizers of logarithmic potentials, see [Sze39, Chapters 5 and 6] and [MOT53, Volume 2, Chapter 10]. Denote

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

the *Hermite polynomials*, and

$$L_n^{(-1)}(x) = \frac{x e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n-1}) \quad ; \quad L_n^{(0)}(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n)$$

the generalized *Laguerre polynomials* of weight  $-1$  and  $0$ . If  $\frac{1}{n} \sum_{i=1}^n \mu_i^2 \leq L$ , then the maximum of  $\Delta(\mu_1, \dots, \mu_n)^2$  is attained if and only if the  $\mu_i$ 's are the  $n$  zeroes of the polynomial

$$H_n \left( \sqrt{\frac{n-1}{2L}} x \right).$$

Similarly, under the same assumption  $\frac{1}{n} \sum_{i=1}^n \mu_i^2 \leq L$ , the maximum of  $(\prod_{i=1}^n \mu_i) \Delta(\mu_1^2, \dots, \mu_n^2)$  is attained if and only if the  $\mu_i$ 's are the  $n$  zeroes of the polynomial  $L_n^{(0)}(nx/L)$ , and the maximum of  $\Delta(\mu_1^2, \dots, \mu_n^2)$  is attained if and only if the  $\mu_i$ 's are the  $n$  zeroes of the polynomial  $L_n^{(-1)}((n-1)x/L)$ . In each case, these values can be calculated, see [Sze39, Theorem 6.71].

This enables one to have an explicit bound on  $T_n(\lambda, t)$  assuming that  $\sum_{i=1}^n \mu_i^2$  is contained in some interval  $[\alpha, \beta]$ . Multiplying these bounds by  $\frac{1}{n!}$  times the number of integer or half-integer points in the “orange peel”

$$\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \in [kn^3, (k+1)n^3] \right\},$$

and then summing these bounds over  $k$  provides an upper bound  $n^{-2\varepsilon}$  on the part of  $S_n(t)$  corresponding to weights such that  $\sum_{i=1}^n \mu_i^2$  is big enough, say bigger than  $10n^3$  — we skip the details of these computations, as they are quite similar to what will be done in §3.2. Unfortunately, we were not able to use this method to bound the whole series  $S_n(t)$ : indeed, for “small” values of  $\sum_{i=1}^n \mu_i^2$ , the points of the lattice of weights cannot approximate sufficiently the roots of the aforementioned renormalized Hermite or Laguerre polynomials, and therefore the corresponding bound was too crude for most of the corresponding terms in  $S_n(t)$ . Nonetheless, this alternative method hints at possible better bounds on  $S_n(t)$ .

**3.1. Guessing the order of decay of the dominating series.** Remember the restriction  $n \geq 2$  (respectively,  $n \geq 3$  and  $n \geq 10$ ) when studying special unitary groups (resp., compact symplectic groups and special orthogonal groups) and their quotients. In what follows, we use the superscript  $\star$  to denote a set of partitions or half-partitions minus the trivial partition  $(0, 0, \dots, 0)$ . The lemma hereafter allows to guess the correct order of decay of the series under study.

**Lemma 17.** *Each weight  $\lambda_{\min}$  indicated in the following table corresponds to an irreducible representation in the case of compact groups, and to a spherical irreducible representation in the case of symmetric spaces of type non-group. The table also gives the corresponding values of  $A_n$  and  $B_n$ . In the group case,  $B_n(\lambda_{\min})$  is minimal among  $\{B_n(\lambda), \lambda \in W_n^\star\}$ .*

$K$ or $G/K$	$\lambda_{\min}$	$B_n(\lambda_{\min})$	$A_n(\lambda_{\min})$
$\mathrm{SO}(2n+1, \mathbb{R})$	$(1, 0, \dots, 0)_n$	$\frac{2n}{2n+1}$	$(2n+1)^2$
$\mathrm{SO}(2n, \mathbb{R})$	$(1, 0, \dots, 0)_n$	$\frac{2n-1}{2n}$	$4n^2$
$\mathrm{SU}(n, \mathbb{C})$	$(1, 0, \dots, 0)_{n-1}$	$1 - \frac{1}{n^2}$	$n^2$
$\mathrm{USp}(n, \mathbb{H})$	$(1, 0, \dots, 0)_n$	$\frac{2n+1}{2n}$	$4n^2$
$\mathrm{Gr}(2n+1, q, \mathbb{R})$	$(2, 0, \dots, 0)_q$	2	$2n^2 + 3n$
$\mathrm{Gr}(2n, q, \mathbb{R})$	$(2, 0, \dots, 0)_q$	2	$2n^2 + n - 1$
$\mathrm{Gr}(n, q, \mathbb{C})$	$(1, 0, \dots, 0)_q$	2	$n^2 - 1$
$\mathrm{Gr}(n, q, \mathbb{H})$	$(1, 1, 0, \dots, 0)_{2q}$	2	$(n-1)(2n+1)$
$\mathrm{SO}(2n, \mathbb{R})/\mathrm{U}(n, \mathbb{C})$	$(1, 1, 0, \dots, 0)_n$	$\frac{2(n-1)}{n}$	$n(2n-1)$
$\mathrm{SU}(n, \mathbb{C})/\mathrm{SO}(n, \mathbb{R})$	$(2, 0, \dots, 0)_{n-1}$	$\frac{2(n-1)(n+2)}{n^2}$	$\frac{n(n+1)}{2}$
$\mathrm{SU}(2n, \mathbb{C})/\mathrm{USp}(n, \mathbb{H})$	$(1, 1, 0, \dots, 0)_{2n-1}$	$\frac{(n-1)(2n+1)}{n^2}$	$n(2n-1)$
$\mathrm{USp}(n, \mathbb{H})/\mathrm{U}(n, \mathbb{C})$	$(2, 0, \dots, 0)_n$	$\frac{2(n+1)}{n}$	$n(2n+1)$

*Remark.* For symmetric spaces of type non-group, one can also check the minimality of  $B_n(\lambda_{\min})$ , except for certain real Grassmannian varieties  $\mathrm{Gr}(n, q, \mathbb{R})$ . For instance, if  $q = 1$ , then  $(1)_q$  labels the geometric representation of  $\mathrm{SO}(n, \mathbb{R})$  on  $\mathbb{C}^n$ , which has indeed an invariant vector by  $\mathrm{SO}(n-1, \mathbb{R}) \times \mathrm{SO}(1, \mathbb{R})$ ; and the corresponding value of  $B(\lambda)$  is  $(n-1)/n < 2$ . Fortunately,  $\lambda_{\min}$ , though not minimal, will still yield in this case the correct order of decay of the series  $S(t)$ .

*Remark.* To each “minimal” weight  $\lambda_{\min}$  corresponds a very natural representation. Namely, for a special orthogonal group  $\mathrm{SO}(n, \mathbb{R})$  (respectively, a compact symplectic group  $\mathrm{USp}(n, \mathbb{H})$ ), the minimizer is the “geometric” representation over  $\mathbb{C}^n$  (respectively  $\mathbb{C}^{2n}$ ) corresponding to the embedding  $\mathrm{SO}(n, \mathbb{R}) \hookrightarrow \mathrm{SO}(n, \mathbb{C}) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$  (respectively  $\mathrm{USp}(n, \mathbb{H}) \hookrightarrow \mathrm{SU}(2n, \mathbb{C}) \hookrightarrow \mathrm{GL}(2n, \mathbb{C})$ ). For a special unitary group  $\mathrm{SU}(n, \mathbb{C})$ , one has again the geometric representation over  $\mathbb{C}^n$ , and its compose with the involution  $k \mapsto (k^t)^{-1}$  corresponds to the label  $(1, \dots, 1)_{n-1}$ , which also minimizes  $B_n(\lambda)$ . The case of spherical minimizers is more involved but still workable: we shall detail it in Section 4.

*Proof.* To avoid any ambiguity, we shall use in the following indices to precise the length of a partition or half-partition. Let us first find the minimizers of  $B_n(\lambda)$  in the group case:

- $\mathrm{SU}(n)$ : one has to minimize

$$-\frac{|\lambda|^2}{n} + \sum_{i=1}^{n-1} \lambda_i^2 + (n+1-2i)\lambda_i = \frac{1}{n} \left( \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2 \right) + \left( \sum_{i=1}^{n-1} i(n-i)(\lambda_i - \lambda_{i+1}) \right) = A + B$$

over  $\mathfrak{Y}_{n-1}^*$ . In  $B$ , at least one term is non-zero, so

$$B \geq \left( \min_{i \in [1, n-1]} i(n-i) \right) = n-1,$$

with equality if and only if  $\lambda = (1, 0, \dots, 0)_{n-1}$  or  $\lambda = (1, \dots, 1)_{n-1}$ . In both cases,  $A$  is then equal to  $\frac{n-1}{n}$ . However,  $\frac{n-1}{n}$  is also the minimum value of  $A$  over  $\mathfrak{Y}_{n-1}^*$ . Indeed, there is at least one index  $l \in \llbracket 1, n-1 \rrbracket$  such that  $\lambda_l > \lambda_{l+1}$ . Then all the  $(\lambda_i - \lambda_j)^2$  with  $i \leq l$  and  $j \geq l+1$  give a contribution at least equal to 1, and there are  $l(n-l)$  such contributions. Thus

$$A \geq \frac{l(n-l)}{n} \geq \frac{n-1}{n},$$

and one concludes that  $\min B_n(\lambda)$  is obtained only for the two aforementioned partitions, and is equal to  $\frac{1}{n}(A_{\min} + B_{\min}) = 1 - \frac{1}{n^2}$ .

- $\text{SO}(2n)$ : the quantity to minimize over  $\frac{1}{2}\mathfrak{Y}_n^*$  is

$$\left( \sum_{i=1}^n \lambda_i^2 \right) + \left( \sum_{i=1}^{n-2} i(2n-1-i)(\lambda_i - \lambda_{i+1}) \right) + n(n-1)\lambda_{n-1} = A + B + C,$$

again with  $A$ ,  $B$  and  $C$  non-negative in each case. Only  $A$  involves  $\lambda_n$ , so a minimizer satisfies necessarily  $\lambda_n = 0$  (partitions) or  $\lambda_n = \frac{1}{2}$  (half-partitions). In the case of partitions, a minimizer of  $B + C$  is  $(1, 0, \dots, 0)_n$ , which gives the value  $\min_{i \in \llbracket 1, n-1 \rrbracket} i(2n-1-i) = 2n-2$ . The same sequence minimizes  $A$  over  $\mathfrak{Y}_n^*$ , so the minimal value of  $A + B + C$  over non-trivial partitions is  $2n-1$  and it is obtained only for  $(1, 0, \dots, 0)_n$ . On the other hand, over half-partitions, the minimizer is  $(\frac{1}{2}, \dots, \frac{1}{2})_n$ , giving the value

$$\frac{n}{4} + \frac{n(n-1)}{2} = \frac{n(2n-1)}{4}.$$

Since we assume  $2n \geq 10$  and therefore  $n \geq 5$ , this value is strictly bigger than  $2n-1$ , so the only minimizer of  $B_n(\lambda)$  in  $\frac{1}{2}\mathfrak{Y}_n^*$  is  $(1, 0, \dots, 0)_n$ .

- $\text{SO}(2n+1)$ : exactly the same reasoning gives the unique minimizer  $(1, 0, \dots, 0)_n$ , with corresponding value  $2n$  for  $A + B + C = (2n+1)B_n(\lambda)$ .
- $\text{USp}(n)$ : here one has only to look at partitions, and the same reasoning as for  $\text{SO}(2n)$  and  $\text{SO}(2n+1)$  yields the unique minimizer  $(1, 0, \dots, 0)_n$ , corresponding to the value  $2n+1$  for  $2nB_n(\lambda)$ .

The spherical minimizers are obtained by the same techniques; however, some cases (with  $n$  or  $q$  too small) are exceptional, so we have only retained in the statement of our Lemma the “generic” minimizer. The corresponding values of  $A_n(\lambda)$  and  $B_n(\lambda)$  are easy calculations.  $\square$

Suppose for a moment that the series  $S_n(t)$  of Proposition 16 has the same behavior as its “largest term”  $A_n(\lambda_{\min})e^{-tB_n(\lambda_{\min})}$ . We shall show in a moment that this is indeed true just after cut-off time (for  $n$  big enough). Then,  $S_n(t)$  is a  $O(\cdot)$  of

$$n^2 e^{-t} \quad \text{for } \text{SU}(n), \text{SO}(n), \text{USp}(n);$$

$$n^2 e^{-2t} \quad \text{for } \text{SU}(n)/\text{SO}(n), \text{SU}(2n)/\text{USp}(n), \text{SO}(2n)/\text{U}(n), \text{USp}(n)/\text{U}(n),$$

$$\text{SU}(n)/\text{S}(\text{U}(n-q) \times \text{U}(q)), \text{SO}(n)/(\text{SO}(n-q) \times \text{SO}(q)), \text{USp}(n)/(\text{USp}(n-q) \times \text{USp}(q)).$$

Set then  $t_{n,\varepsilon} = \alpha(1+\varepsilon)\log n$ , with  $\alpha = 2$  in the case  $n^2 e^{-t}$ , and  $\alpha = 1$  in the case  $n^2 e^{-2t}$ . Under the assumption  $S_n(t) \sim A_n(\lambda_{\min})e^{-tB_n(\lambda_{\min})}$ , one has

$$S_n(t_{n,\varepsilon}) = O(n^{-2\varepsilon}).$$

Thus, the previous computations lead to the following guess: the cut-off time is

- $2\log n$  for classical simple compact Lie groups;
- $\log n$  for classical simple compact symmetric spaces of type non-group.

**3.2. Growth of the dimensions versus decay of the Laplace-Beltrami eigenvalues.** The estimate  $S_n(t_{n,\varepsilon}) \sim A_n(\lambda_{\min}) e^{-t_{n,\varepsilon} B_n(\lambda)} = O(n^{-2\varepsilon})$  might seem very optimistic; nonetheless, we are going to prove that the sum of all the other terms  $A_n(\lambda) e^{-t_{n,\varepsilon} B_n(\lambda)}$  in  $S_n(t)$  does not change too much this bound, and that one still has at least

$$S(t_{n,\varepsilon}) = O(n^{-\frac{\varepsilon}{2}}).$$

We actually believe that at least in the group case, the exponent  $2\varepsilon$  is good, *cf.* the remark before §3.1 — the previous discussion shows that it is then optimal.

Suppose that one can bound  $A_n(\lambda) e^{-t_{n,\varepsilon} B_n(\lambda)}$  by  $c(n)^{|\lambda|}$ , where  $|\lambda|$  is the size of the partition (sum of all parts) and  $c(n)$  is some function of  $n$  that goes to 0 as  $n$  goes to infinity (say,  $Cn^{-\delta\varepsilon}$ ). We can then use the following:

**Lemma 18.** *Suppose  $x \leq \frac{1}{2}$ . Then, the sum over all partitions  $\sum_{\lambda} x^{|\lambda|}$ , which is convergent, is smaller than  $1 + 5x$ . Consequently,*

$$\sum_{\substack{\lambda \in \mathfrak{Y}_n \\ \lambda \neq (0, \dots, 0)}} x^{|\lambda|} \leq 5x.$$

*Proof.* The power series  $P(x) = \sum_{\lambda} x^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots$  has radius of convergence 1, and it is obviously convex on  $\mathbb{R}_+$ . Thus, it suffices to verify the bound at  $x = 0$  and  $x = \frac{1}{2}$ . However,

$$P(0) = 1 = 1 + (5 \times 0) \quad ; \quad P\left(\frac{1}{2}\right) \leq 3.463 \leq 1 + \left(5 \times \frac{1}{2}\right).$$

□

With this in mind, the idea is then to control the growth<sup>9</sup> of the coefficients  $A_n(\lambda)$ , starting from the trivial partition  $(0, \dots, 0)$ . Of course, the arguments will be slightly different in each case, and we will have to adapt them to the case of half-partitions, even partitions, partitions with parts doubled, *etc.*; however, the reasoning will be the same each time. The simplest cases to treat in this perspective are the compact symplectic groups and their quotients.

**3.2.1. Symplectic groups and their quotients.** Set  $t_{n,\varepsilon} = 2(1 + \varepsilon) \log n$ ; in particular,  $t_{n,0} = 2 \log n$ . We fix a partition  $\lambda \in \mathfrak{Y}_n$ , and for  $k \leq \lambda_n$ , we denote  $\rho_{k,n}$  the quotient of the dimensions  $D^\lambda$  associated to the two rectangular partitions

$$(k, \dots, k)_n \quad \text{and} \quad (k-1, \dots, k-1)_n. \quad (3.1)$$

Using the formula given in §2.3 in the case of compact symplectic groups, one obtains:

$$\rho_{k,n} = \prod_{1 \leq i \leq j \leq n} \frac{2k + 2n + 2 - i - j}{2k + 2n - i - j} = \prod_{1 \leq i \leq j \leq n} 1 + \frac{2}{2k + 2n - i - j} \leq \exp \left( \sum_{1 \leq i \leq j \leq n} \frac{2}{2k + 2n - i - j} \right).$$

The double sum can be estimated by standard comparison techniques between sums and integrals. Namely, since  $x, y \mapsto \frac{1}{2k + 2n - x - y}$  is convex on  $\{(x, y) \mid x \geq 0, y \geq 0, 2k + 2n \geq x + y\}$ , one can bound each term by

$$\frac{2}{2k + 2n - i - j} \leq \iint_{[i-\frac{1}{2}, i+\frac{1}{2}] \times [j-\frac{1}{2}, j+\frac{1}{2}]} \frac{2}{2k + 2n - x - y} dx dy.$$

We use this bound for non-diagonal terms with indices  $i < j$ , and for diagonal terms with  $i = j$ , we use the simpler bound

$$\sum_{i=1}^n \frac{1}{k + n - i} = \sum_{u=0}^{n-1} \frac{1}{k + u} = H_{k+n-1} - H_{k-1} \leq \frac{1}{k} + \log(k + n - 1) - \log k$$

<sup>9</sup>This is also what is done in [Por96a, Por96b], but the way we make our partitions grow is different.

where  $H_n$  denotes the  $n$ -th harmonic sum. So,

$$\begin{aligned} \log \rho_{k,n} &\leq \sum_{1 \leq i \leq j \leq n} \frac{2}{2k+2n-i-j} \leq H_{k+n-1} - H_{k-1} + \iint_{[\frac{1}{2}, n+\frac{1}{2}]^2} \frac{1}{2k+2n-x-y} dx dy \\ &\leq \frac{1}{k} + \log(k+n-1) - \log k \\ &\quad + (2k+2n-1) \log(2k+2n-1) + (2k-1) \log(2k-1) - 2(2k+n-1) \log(2k+n-1). \end{aligned}$$

On the other hand, the same transformation on partitions makes  $-t_{n,0} B_n(\lambda)$  evolve by  $-(2k+n) \log n$ . So, if  $\eta_{k,n}^2$  is the quotient of the quantities  $(D^\lambda)^2 e^{-t_{n,0} B_n(\lambda)}$  with  $\lambda$  as in Equation (3.1), then

$$\begin{aligned} \log \eta_{k,n} &\leq -\frac{2k+n}{2} \log n + \frac{1}{k} + \log(k+n-1) - \log k \\ &\quad + (2k+2n-1) \log(2k+2n-1) + (2k-1) \log(2k-1) - 2(2k+n-1) \log(2k+n-1). \end{aligned}$$

Suppose  $k \geq 2$ . Then, one can fix  $n \geq 3$  and study the previous expression as a function of  $k$ . Its derivative is then always negative, so  $\log \eta_{k,n} \leq \log \eta_{2,n}$ , which is also always negative. From this, one deduces that

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} \leq \eta_{1,n}$$

for any rectangular partition  $(\lambda_n, \dots, \lambda_n)_n$ ; indeed, the left-hand side is the product of the contributions  $\eta_{k,n}$  for  $k$  in  $\llbracket 1, \lambda_n \rrbracket$ . However,  $\eta_{1,n}$  is also smaller than 1: in this case, the dimension is given by the exact formula

$$D^{(1, \dots, 1)_n} = \text{Cat}_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1},$$

so  $\eta_{1,n} = \text{Cat}_{n+1} e^{-\frac{n+2}{2} \log n}$ , which can be checked to be smaller than 1 for every  $n \geq 3$ . So in fact,

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} \leq 1$$

for any rectangular partition  $(\lambda_n, \dots, \lambda_n)_n$ .

The previous discussion hints at the following more general result:

**Proposition 19.** *In the case of compact symplectic groups, at cut-off time,*

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} \leq \frac{14}{3}$$

*for any integer partition  $\lambda$  of length  $n$  (not only the rectangular partitions).*

*Proof.* We fix  $l \in \llbracket 1, n-1 \rrbracket$ , and the idea is again to study the quotient  $\rho_{k,l}$  of the dimensions associated to the two partitions

$$(k + \lambda_{l+1}, \dots, k + \lambda_{l+1}, \lambda_{l+1}, \dots, \lambda_n)_n \quad \text{and} \quad (k-1 + \lambda_{l+1}, \dots, k-1 + \lambda_{l+1}, \lambda_{l+1}, \dots, \lambda_n)_n, \quad (3.2)$$

where  $k$  is some integer smaller than  $\lambda_l - \lambda_{l+1}$  — in other words, the  $n-l$  last parts of our partition have already been constructed, and one adds  $k$  to the  $l$  first parts, until  $k = \lambda_l - \lambda_{l+1}$ ; see Figure 4.

The transformation on partitions described by Equation (3.2) makes the quantity  $-t_{n,0} B_n(\lambda)$  change by  $-\frac{l(2k'+2n-l)}{n} \log n$ . We shall prove in the following that this variation plus  $\log \rho_{k,l}$  is almost always negative. For convenience, we will treat separately the cases  $l = 1$  or  $2$  and the case  $l \geq 3$ ; hence, suppose first that  $l \in \llbracket 3, n-1 \rrbracket$ . The quotients of Vandermonde determinants can be simplified as follows:

$$\rho_{k,l} = \prod_{j=l+1}^n \frac{k+j-1+\lambda_{l+1}-\lambda_j}{k+j-l-1+\lambda_{l+1}-\lambda_j} \frac{k+\lambda_{l+1}+\lambda_j+2n+1-j}{k+\lambda_{l+1}+\lambda_j+2n+1-j-l} \prod_{1 \leq i \leq j \leq l} \frac{2k+2\lambda_{l+1}+2n+2-i-j}{2k+2\lambda_{l+1}+2n-i-j}.$$



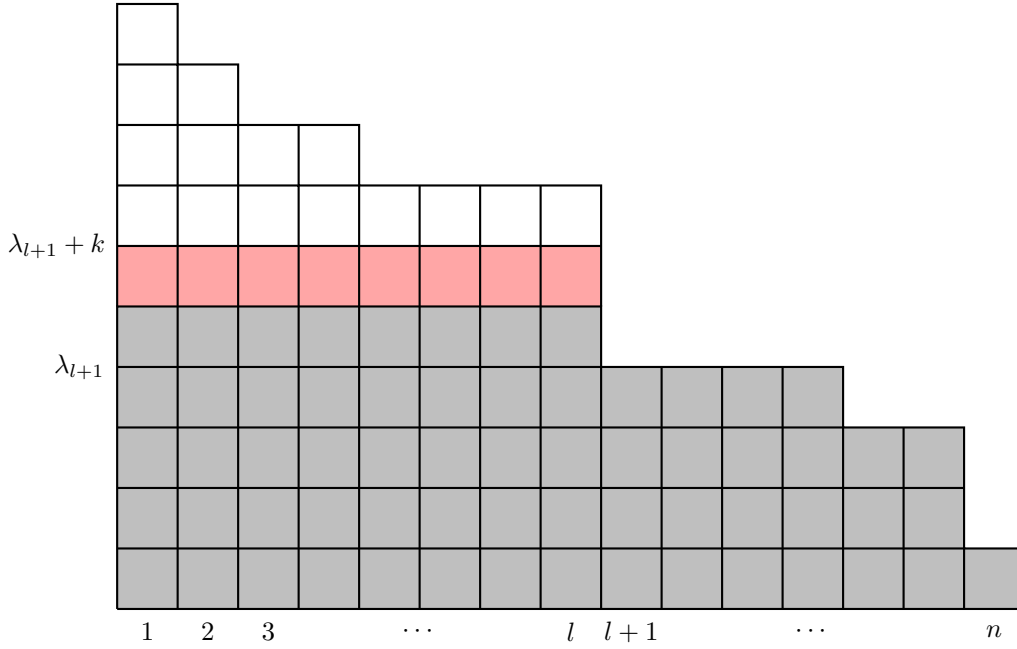


FIGURE 4. One makes the partitions grow layer by layer, starting from the bottom.

Notice that the second product  $\rho_{k,l,(2)}$  in this formula is very similar to  $\rho_{k,n}$ ; the main difference is that indices  $i, j$  are now smaller than  $l$  (instead of  $n$ ). Hence, by adapting the arguments, one obtains

$$\begin{aligned} \log \rho_{k,l,(2)} &\leq \sum_{1 \leq i \leq j \leq l} \frac{2}{2k' + 2n - i - j} \leq H_{k'+n-1} - H_{k'+n-l-1} + \iint_{[\frac{1}{2}, l+\frac{1}{2}]^2} \frac{1}{2k' + 2n - x - y} dx dy \\ &\leq \frac{1}{k' + n - l} + \log(k' + n - 1) - \log(k' + n - l) + (2k' + 2n - 1) \log(2k' + 2n - 1) \\ &\quad + (2k' + 2n - 2l - 1) \log(2k' + 2n - 2l - 1) - 2(2k' + 2n - l - 1) \log(2k' + 2n - l - 1) \end{aligned}$$

where  $k'$  stands for  $k + \lambda_{l+1}$ . So, if  $(\eta_{k,l})^2$  is the quotient of the quantities  $(D^\lambda)^2 e^{-t_{n,0} B_n(\lambda)}$  with  $\lambda$  as in Equation (3.2), then  $\log \eta_{k,l} \leq \log \tilde{\eta}_{k,l} + \log \rho_{k,l,(1)}$ , where  $\log \tilde{\eta}_{k,l}$  is given by

$$\begin{aligned} &-\frac{l(2k' + 2n - l)}{2n} \log n + \frac{1}{k' + n - l} + \log(k' + n - 1) - \log(k' + n - l) \\ &+ (2k' + 2n - 1) \log(2k' + 2n - 1) + (2k' + 2n - 2l - 1) \log(2k' + 2n - 2l - 1) \\ &- 2(2k' + 2n - l - 1) \log(2k' + 2n - l - 1), \end{aligned}$$

and  $\rho_{k,l,(1)}$  is the first product in the expansion of  $\rho_{k,l}$ . Let us analyze these two quantities separately.

- $\log \tilde{\eta}_{k,l}$ : here the technique is really the same as for  $\log \eta_{k,n}$ . Namely, with  $n$  and  $l$  fixed,  $\log \tilde{\eta}_{k,l}$  appears as a decreasing function of  $x = k'$ , because its derivative with respect to  $x$  is

$$\begin{aligned} &-\frac{l \log n}{n} - \frac{1}{(x + n - l)^2} + \frac{1}{x + n - 1} - \frac{1}{x + n - l} \\ &+ 2(\log(2x + 2n - 1) + \log(2x + 2n - 2l - 1) - 2 \log(2x + 2n - l - 1)). \end{aligned}$$

A upper bound on the first line is  $-\frac{(l-1) \log n}{n} \leq 0$  (remember that  $n \geq 3$  and therefore  $\log n \geq 1$ ), and the second line is negative by concavity of the logarithm. From this, one deduces that  $\log \tilde{\eta}_{k,l} \leq \log \tilde{\eta}_{1,l}$ , and we shall use this estimate in the following in order to compensate the other

part of  $\log \eta_{k,l}$ :

$$\begin{aligned} \log \tilde{\eta}_{k,l} \leq & -\frac{l(2v+2+2n-l)}{2n} \log n + \frac{1}{v+n+1-l} + \log(v+n) - \log(v+n+1-l) \\ & + (2v+2n+1) \log(2v+2n+1) + (2v+2n-2l+1) \log(2v+2n-2l+1) \\ & - 2(2v+2n-l+1) \log(2v+2n-l+1) \end{aligned}$$

where  $v$  stands for  $\lambda_{l+1}$ .

- $\log \rho_{k,l,(1)}$ : in the product  $\rho_{k,l,(1)}$ , each term of index  $j$  writes as

$$\begin{aligned} \frac{(k'+n)^2 - (\lambda_j + n + 1 - j)^2}{(k' + n - l)^2 - (\lambda_j + n + 1 - j)^2} & \leq \frac{(k' + n)^2 - (\lambda_{l+1} + n + 1 - j)^2}{(k' + n - l)^2 - (\lambda_{l+1} + n + 1 - j)^2} \\ & \leq \frac{k+j-1}{k+j-l-1} \frac{k''+2n+1-j}{k''+2n+1-j-l} \end{aligned}$$

with  $k'' = k + 2\lambda_{l+1} = k + 2v$ ; and multiplying all these bounds together, one gets

$$\rho_{k,l,(1)} \leq \frac{(k+n-1)!}{(k+l-1)!} \frac{(k-1)!}{(k+n-l-1)!} \frac{(k''+2n-l)!}{(k''+n)!} \frac{(k''+n-l)!}{(k''+2n-2l)!}.$$

Again, this is decreasing in  $k$ , so

$$\rho_{k,l,(1)} \leq \frac{n!(2v+2n-l+1)!(2v+n-l+1)!}{l!(n-l)!(2v+n+1)!(2v+2n-2l+1)!}.$$

Recall the classical Stirling estimates: for  $m \geq 1$ ,

$$\log m! = m \log m + \frac{1}{2} \log m - m + \log \sqrt{2\pi} + \frac{1}{12m} - r_m, \quad \text{with } 0 \leq r_m \leq \frac{1}{360m^3}.$$

It enables us to bound  $\log \rho_{k,l,(1)}$  by the sum of the following quantities:

- ★  $A = (2v+2n-l+1) \log(2v+2n-l+1) + (2v+n-l+1) \log(2v+n-l+1) - (2v+n+1) \log(2v+n+1) - (2v+2n-2l+1) \log(2v+2n-2l+1).$
- ★  $B = \frac{1}{2}(\log(2v+2n-l+1) + \log(2v+n-l+1) - \log(2v+n+1) - \log(2v+2n-2l+1)),$  which is non-positive by concavity of the logarithm.
- ★  $C = n \log n - l \log l - (n-l) \log(n-l).$
- ★  $D = \frac{1}{2}(\log n - \log l - \log(n-l)).$  This is non-positive unless  $n = l+1$  — recall that we assume for the moment  $l \in \llbracket 3, n-1 \rrbracket$ . In that case, it is smaller than  $\frac{1}{2(n-1)}.$
- ★  $E = \frac{1}{12} \left( \frac{1}{n} - \frac{1}{l} - \frac{1}{n-l} + \frac{1}{2v+2n-l+1} + \frac{1}{2v+n-l+1} - \frac{1}{2v+n+1} - \frac{1}{2v+2n-2l+1} \right).$
- ★  $F = \frac{1}{360} \left( \frac{1}{l^3} + \frac{1}{(n-l)^3} + \frac{1}{(2v+n+1)^3} + \frac{1}{(2v+2n-2l+1)^3} \right).$

The sum of the two last terms  $EF = E + F$  happens to be negative. Indeed,  $E$  and  $F$  are decreasing in  $v$  (we use the convexity of  $x \mapsto \frac{1}{x^2}$  to show that  $\frac{dE}{dv} \leq 0$ ), so it suffices to check the result when  $v = 0$ . Then, with  $l$  fixed,

$$\begin{aligned} EF(n, l) = & \frac{1}{12} \left( \frac{1}{n} - \frac{1}{l} - \frac{1}{n-l} + \frac{1}{2n-l+1} + \frac{1}{n-l+1} - \frac{1}{n+1} - \frac{1}{2n-2l+1} \right) \\ & + \frac{1}{360} \left( \frac{1}{l^3} + \frac{1}{(n-l)^3} + \frac{1}{(n+1)^3} + \frac{1}{(2n-2l+1)^3} \right) \end{aligned}$$

is decreasing in  $n$ , hence smaller than its value when  $n = l+1$ . So, it suffices to look at  $EF(l+1, l)$ , which is now increasing in  $l$ , but still negative. Thus, in the following, we shall use the bound

$$\log \rho_{k,l,(1)} \leq A + C + D \leq A + C + \frac{1}{2n-2}.$$

Adding together the bounds previously demonstrated, we get

$$\begin{aligned} \log \eta_{k,l} \leq & -\frac{l(2v+2+2n-l)}{2n} \log n + \frac{1}{2n-2} + \frac{1}{v+n+1-l} + \log(v+n) - \log(v+n+1-l) \\ & + (2v+2n+1) \log(2v+2n+1) - (2v+2n-l+1) \log(2v+2n-l+1) \\ & + (2v+n-l+1) \log(2v+n-l+1) - (2v+n+1) \log(2v+n+1) \\ & + n \log n - l \log l - (n-l) \log(n-l). \end{aligned}$$

By concavity of  $x \log x$ , the sum of the second and third rows is non-positive. What remains is decreasing in  $l$  and in  $v$ , and when  $l = 3$  and  $v = 0$ , we get

$$\frac{3}{2n} \log n + \frac{1}{2n-2} + \frac{1}{n-2} + \log\left(\frac{n}{n-2}\right) + (n-3) \log\left(\frac{n}{n-3}\right) - 3 \log 3$$

which is maximal for  $n = 5$ , and still (barely) negative at this value. Thus, we have shown so far that  $\eta_{k,l} \leq 1$  for any  $k$ , any  $l \in \llbracket 3, n-1 \rrbracket$ , and any partition  $\lambda$  that we fill as in Figure 4.

When  $l = 1$  or  $l = 2$ , the approximations on  $\log \eta_{k,l}$  that we were using before are not good enough, but we can treat these cases separately. When  $l = 1$ ,

$$\begin{aligned} \rho_{k,1} &= \frac{\lambda_2 + k + n}{\lambda_2 + k + n - 1} \prod_{j=2}^n \frac{k+j-1+\lambda_2-\lambda_j}{k+j-2+\lambda_2-\lambda_j} \frac{k+\lambda_2+\lambda_j+2n+1-j}{k+\lambda_2+\lambda_j+2n-j} \\ &\leq \frac{k+n}{k+n-1} \prod_{j=2}^n \frac{k+j-1}{k+j-2} \frac{k+2n+1-j}{k+2n-j} = \frac{k+2n-1}{k}; \\ \eta_{k,1} &\leq \frac{k+2n-1}{k} e^{-\frac{2k+2n-1}{2n} \log n}. \end{aligned}$$

If  $k = 1$ , which only happens once when one makes the partition grow, then the bound above is  $2n e^{-\frac{2n+1}{2n} \log n} \leq 2$ . On the other hand, if  $k \geq 2$ , then the bound is decreasing in  $k$  and therefore smaller than  $(n + \frac{1}{2}) e^{-\frac{2n+3}{2n} \log n} \leq 1$ . So, one also has  $\eta_{k,1} \leq 1$  for any  $k$  but  $k = 1$ , where a correct bound is 2. Similarly, when  $l = 2$ ,

$$\begin{aligned} \rho_{k,2} &= \frac{\lambda_3 + k + n}{\lambda_3 + k + n - 2} \frac{2\lambda_3 + 2k + 2n - 1}{2\lambda_3 + 2k + 2n - 3} \prod_{j=3}^n \frac{k+j-1+\lambda_3-\lambda_j}{k+j-3+\lambda_3-\lambda_j} \frac{k+\lambda_3+\lambda_j+2n+1-j}{k+\lambda_3+\lambda_j+2n-1-j} \\ &\leq \frac{k+n}{k+n-2} \frac{2k+2n-1}{2k+2n-3} \prod_{j=3}^n \frac{k+j-1}{k+j-3} \frac{k+2n+1-j}{k+2n-1-j} = \frac{k+2n-2}{k} \frac{k+2n-1}{k+1} \frac{2k+2n-1}{2k+2n-3}; \\ \eta_{k,2} &\leq \frac{k+2n-2}{k} \frac{k+2n-1}{k+1} \frac{2k+2n-1}{2k+2n-3} e^{-\frac{2n+2k-2}{n} \log n}. \end{aligned}$$

Again, the last bound is decreasing in  $k$ , smaller than  $2 + \frac{1}{n} \leq \frac{7}{3}$  when  $k = 1$  and smaller than 1 when  $k = 2$ . Hence,  $\eta_{k,2} \leq 1$  unless  $k = 1$ , where a correct bound is  $\frac{7}{3}$  (and again this situation occurs at most once whence making the partition grow).

Conclusion: every quotient  $\eta_{k,l}$  satisfies  $\eta_{k,l} \leq 1$ , but the two following exceptions:  $k = 1$  and  $l = 1$  or 2. The product of the bounds on these two exceptions is  $2 \times \frac{7}{3} = \frac{14}{3}$ , so for every partition  $\lambda$ , one has indeed

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} = \prod_{l=1}^n \prod_{k=1}^{\lambda_l - \lambda_{l+1}} \eta_{k,l} \leq \frac{14}{3}.$$

□

*Remark.* A small refinement of the previous proof shows that the worst case is in fact the partition  $(2, 1, 0, \dots, 0)_n$  — by that we mean that any other partition has quotients  $\rho_{k,l}$  that are smaller. Its dimension is provided by the exact formula

$$D^\lambda = \frac{8n(n^2 - 1)}{3},$$

so one can replace the bound  $\frac{14}{3}$  of Proposition 19 by  $\frac{8}{3}$ .

*Remark.* The main trick in the proof of Proposition 19 is the way we make our partitions grow. One could have tried something simpler, namely, make the first part  $\lambda_1$  grow box by box, then the second part  $\lambda_2$ , etc. However, with the same technique of estimation of quotients of dimensions, we were then not able to prove something better than

$$D^\lambda \leq C e^{-t_{n,0} B_n(\lambda)},$$

instead of  $C e^{-\frac{t_{n,0}}{2} B_n(\lambda)}$  — this is in accordance with the bound  $2\alpha \log n$  mentioned in Saloff-Coste's Theorem 7. By making the partitions grow layer by layer, we use in a much better way the fact that the parts  $\lambda_1, \lambda_2, \dots, \lambda_n$  are ordered, and the “compensations” in the growth of  $D^\lambda$  given by the determinantal structure of its formula.

The upper bound (1.11) is now an easy consequence of Lemma 18 and Proposition 19. For any partition  $\lambda$ , notice that

$$B_n(\lambda) \geq \frac{1}{2n} \sum_{i=1}^n (2n + 2 - 2i) \lambda_i = \frac{1}{2n} \sum_{i=1}^n i(2n + 1 - i)(\lambda_i - \lambda_{i+1}) \geq \frac{1}{2} \sum_{i=1}^n i(\lambda_i - \lambda_{i+1}) = \frac{|\lambda|}{2}.$$

From this, one deduces that in the case of compact symplectic groups,

$$S_n(t_{n,\varepsilon}) = \sum_{\lambda \in \mathfrak{Y}_n^*} (D^\lambda)^2 e^{-t_{n,\varepsilon} B_n(\lambda)} \leq \frac{64}{9} \sum_{\lambda \in \mathfrak{Y}_n^*} e^{-\varepsilon |\lambda| \log n} \leq \frac{320}{9n^\varepsilon} \leq \frac{36}{n^\varepsilon}$$

if one assumes that  $\frac{1}{n^\varepsilon} \leq \frac{1}{2}$  (in order to apply Lemma 18). By Proposition 16, one concludes that

$$d_{\text{TV}}^{\text{USp}(n, \mathbb{H})}(\mu_{2(1+\varepsilon) \log n}, \text{Haar}) \leq \frac{3}{n^{\frac{\varepsilon}{2}}}.$$

Here one can remove the assumption  $\frac{1}{n^\varepsilon} \leq \frac{1}{2}$ : otherwise, the right-hand side is bigger than 1 and therefore the inequality is trivially satisfied. This ends the proof of the upper bound in the case of compact symplectic groups. For their quotients, one can still use Proposition 19, as follows. For quaternionic Grassmannians,

$$S_n\left(\frac{t_{n,\varepsilon}}{2}\right) = \sum_{\lambda \in \mathfrak{Y}_{2q}^*} D^\lambda e^{-\frac{t_{n,\varepsilon}}{2} B_n(\lambda)} \leq \frac{8}{3} \sum_{\lambda \in \mathfrak{Y}_n^*} e^{-\frac{\varepsilon}{2} |\lambda| \log n} \leq \frac{40}{3n^{\frac{\varepsilon}{2}}} \leq \frac{16}{n^{\frac{\varepsilon}{2}}}$$

assuming  $\frac{1}{n^{\frac{\varepsilon}{2}}} \leq \frac{1}{2}$ . This implies that

$$d_{\text{TV}}^{\text{Gr}(n, q, \mathbb{H})}(\mu_{(1+\varepsilon) \log n}, \text{Haar}) \leq \frac{2}{n^{\frac{\varepsilon}{4}}}.$$

Again, the assumption on  $n^{\frac{\varepsilon}{2}}$  is superfluous, since otherwise the right-hand side is bigger than 1. Exactly the same proof works for the spaces  $\text{USp}(n)/\text{U}(n)$ , with the same bound (it may be improved by using the fact that one looks only at even partitions).

**3.2.2. Odd special orthogonal groups and their quotients.** Though the same reasoning holds in every case, we unfortunately have to check case by case that everything works. For odd special orthogonal groups  $\mathrm{SO}(2n+1, \mathbb{R})$ , set  $t_{n,\varepsilon} = 2(1+\varepsilon)\log(2n+1)$ , with in particular  $t_{n,0} = 2\log(2n+1)$ . The main difference between  $\mathrm{SO}(2n+1)$  and  $\mathrm{USp}(n)$  is the appearance of half-partitions, which is solved by the following Lemma:

**Lemma 20.** *For any integer partition  $\lambda$ , denote  $\lambda \boxplus \frac{1}{2}$  the half-partition  $\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \dots, \lambda_n + \frac{1}{2}$ .*

$$\frac{D^{\lambda \boxplus \frac{1}{2}}}{D^\lambda} e^{-\frac{t_{n,0}}{2}(B_n(\lambda \boxplus \frac{1}{2}) - B_n(\lambda))} \leq e^{n(\log 2 - \frac{\log(2n+1)}{4})} \leq 2.$$

*Proof.* The quotient of dimensions is

$$\prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{\lambda_i + \lambda_j + 2n + 1 - i - j} \leq \prod_{1 \leq i \leq j \leq n} \frac{2n + 2 - i - j}{2n + 1 - i - j} = 2^n,$$

and the difference  $\frac{t_{n,0}}{2}(B_n(\lambda \boxplus \frac{1}{2}) - B_n(\lambda))$  is equal to

$$\frac{\log(2n+1)}{2n+1} \left( \sum_{i=1}^n \lambda_i + \frac{1}{4} + \frac{2n+1-2i}{2} \right) \geq \frac{\log(2n+1)}{2n+1} \left( \frac{n}{4} + \frac{n^2}{2} \right) = \frac{n \log(2n+1)}{4}.$$

This yields the first part of the inequality, and the second part is an easy analysis of the variations of the bound with respect to  $n$ .  $\square$

Then, for any integer partition  $\lambda$ , one can as before prove a uniform bound on  $D^\lambda e^{-\log(2n+1) B_n(\lambda)}$ ; the differences are tiny, *e.g.*, in many formulas,  $2n+2$  is replaced by  $2n+1$ , or  $\frac{1}{2n}$  is replaced by  $\frac{1}{2n+1}$ . So more precisely, with the same schema of growth of partitions, one has the following bounds:

- $\eta_{1,n}$ : it is given by the exact formula  $\binom{2n+1}{n+1} e^{-\frac{n(n+1)}{2n+1} \log(2n+1)}$ , which is indeed smaller than 1 for  $n \geq 5$ .
- $\eta_{k \geq 2,n}$ : the comparison techniques between sums and integrals give

$$\begin{aligned} \log \eta_{k,n} &\leq -\frac{n(2k-1+n)}{2n+1} \log(2n+1) + \frac{2}{2k-1} + \frac{1}{k} + \log(k+n-2) - \log k \\ &\quad + (2k+2n-2) \log(2k+2n-2) + (2k-2) \log(2k-2) - 2(2k+n-2) \log(2k+n-2). \end{aligned}$$

This bound is decreasing in  $k$ , whence smaller than its value when  $k=2$ , which is negative for every value of  $n \geq 5$ .

- $\eta_{k,l \in \llbracket 3, n-1 \rrbracket}$ : as before,  $\rho_{k,l}$  splits into  $\rho_{k,l,(1)}$  and  $\rho_{k,l,(2)}$ :

$$\rho_{k,l} = \prod_{j=l+1}^n \frac{k+j-1+\lambda_{l+1}-\lambda_j}{k+j-l-1+\lambda_{l+1}-\lambda_j} \frac{k+\lambda_{l+1}+\lambda_j+2n-j}{k+\lambda_{l+1}+\lambda_j+2n-j-l} \prod_{1 \leq i \leq j \leq l} \frac{2k+2\lambda_{l+1}+2n+1-i-j}{2k+2\lambda_{l+1}+2n-1-i-j}.$$

The bound on  $\log \tilde{\eta}_{k,l}$ , the sum of  $\log \rho_{k,l,(2)}$  and of the variation of  $-\frac{t_{n,0}}{2} B_n(\lambda)$ , is

$$\begin{aligned} \log \tilde{\eta}_{k,l} &\leq -\frac{l(2k'-1+2n-l)}{2n+1} \log(2n+1) + \frac{1}{k'+n-l-1} + \log(k'+n-2) - \log(k'+n-l-1) \\ &\quad + (2k'+2n-2) \log(2k'+2n-2) + (2k'+2n-2l-2) \log(2k'+2n-2l-2) \\ &\quad - 2(2k'+2n-l-2) \log(2k'+2n-l-2) \\ &\leq -\frac{l(2v+2n+1-l)}{2n+1} \log(2n+1) + \frac{1}{v+n-l} + \log(v+n-1) - \log(v+n-l) \\ &\quad + (2v+2n) \log(2v+2n) + (2v+2n-2l) \log(2v+2n-2l) \\ &\quad - 2(2v+2n-l) \log(2v+2n-l) \end{aligned}$$

with  $k' = k + \lambda_{l+1} = k + v$ . On the other hand, in the product  $\rho_{k,l,(1)}$ , each term of index  $j$  writes as

$$\begin{aligned} \frac{(k' + n - 1/2)^2 - (\lambda_j + n + 1/2 - j)^2}{(k' + n - 1/2 - l)^2 - (\lambda_j + n + 1/2 - j)^2} &\leq \frac{(k' + n - 1/2)^2 - (\lambda_{l+1} + n + 1/2 - j)^2}{(k' + n - 1/2 - l)^2 - (\lambda_{l+1} + n + 1/2 - j)^2} \\ &\leq \frac{k + j - 1}{k + j - l - 1} \frac{k'' + 2n - j}{k'' + 2n - j - l}, \end{aligned}$$

so the quantity  $\rho_{k,l,(1)}$  is bounded by

$$\begin{aligned} &\frac{(k + n - 1)!}{(k + l - 1)!} \frac{(k - 1)!}{(k + n - l - 1)!} \frac{(k'' + 2n - l - 1)!}{(k'' + n - 1)!} \frac{(k'' + n - l - 1)!}{(k'' + 2n - 2l - 1)!} \\ &\leq \frac{n! (2v + 2n - l)! (2v + n - l)!}{l! (n - l)! (2v + n)! (2v + 2n - 2l)!}. \end{aligned}$$

Again, Stirling approximation leads to

$$\begin{aligned} \log \rho_{k,l,(1)} &\leq (2v + 2n - l) \log(2v + 2n - l) + (2v + n - l) \log(2v + n - l) - (2v + n) \log(2v + n) \\ &\quad - (2v + 2n - 2l) \log(2v + 2n - 2l) + n \log n - l \log l - (n - l) \log(n - l) + \frac{1}{2n - 2}, \end{aligned}$$

and therefore

$$\begin{aligned} \log \eta_{k,l} &\leq -\frac{l(2v + 2n + 1 - l)}{2n + 1} \log(2n + 1) + \frac{1}{v + n - l} + \log(v + n - 1) - \log(v + n - l) \\ &\quad + (2v + 2n) \log(2v + 2n) + (2v + n - l) \log(2v + n - l) \\ &\quad - (2v + n) \log(2v + n) - (2v + 2n - l) \log(2v + 2n - l) \\ &\quad + n \log n - l \log l - (n - l) \log(n - l) + \frac{1}{2n - 2} \\ &\leq -\frac{l(2n + 1 - l)}{2n + 1} \log(2n + 1) + \frac{1}{n - l} + \log(n - 1) - \log(n - l) \\ &\quad + n \log n - l \log l - (n - l) \log(n - l) + \frac{1}{2n - 2}. \end{aligned}$$

The last bound is decreasing in  $l$ , so it suffices to look at the case  $l = 3$ ; then the bound is decreasing in  $n$ , so it suffices to check that the bound is negative when  $n = 5$ , which is just a computation. We conclude that  $\log \eta_{k,l} \leq 0$  for any  $k$  and any  $l \in \llbracket 3, n - 1 \rrbracket$ .

- $\eta_{k,1}$ : a bound on  $\rho_{k,1}$  is  $\frac{k+n-2}{k} \frac{2k+2n-1}{2k+2n-3}$ , so

$$\begin{aligned} \eta_{k,1} &\leq \frac{k + n - 2}{k} \frac{2k + 2n - 1}{2k + 2n - 3} e^{-\frac{2k+2n-2}{2n+1} \log(2n+1)} \leq \frac{(n-1)(2n+1)}{2n-1} e^{-\frac{2n}{2n+1} \log(2n+1)} \\ &\leq \frac{n-1}{2n-1} e^{\frac{\log(2n+1)}{2n+1}} \leq \frac{1}{2} e^{\frac{\log 11}{11}} \leq 1. \end{aligned}$$

- $\eta_{k,2}$ : a bound on  $\rho_{k,2}$  is  $\frac{k+2n-4}{k} \frac{k+2n-3}{k+1} \frac{2k+2n-1}{2k+2n-5} \frac{k+n-1}{k+n-2}$ , so

$$\eta_{k,2} \leq \frac{k + 2n - 4}{k} \frac{k + 2n - 3}{k + 1} \frac{2k + 2n - 1}{2k + 2n - 5} \frac{k + n - 1}{k + n - 2} e^{-\frac{4k+4n-6}{2n+1} \log(2n+1)} \leq \frac{n}{2n+1} e^{\frac{4 \log(2n+1)}{2n+1}}.$$

The last bound is bigger than 1 only when  $n = 5$  or 6. The maximal value is obtained for  $n = 5$ , and is smaller than  $1.09 \leq \frac{11}{10}$ . Moreover, if  $k \geq 2$ , then one has a much better bound, that is smaller than 1 even when  $n = 5$  or 6.

Putting all together, one sees that at most one quotient  $\eta_{k,l}$  may be bigger than 1 (and actually only when  $n = 5$  or 6). Thus, we have proved the following:

**Proposition 21.** *In the case of odd special orthogonal groups, at cut-off time,*

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} \leq \frac{11}{10}$$

for any integer partition  $\lambda$  of length  $n$ . For half-integer partitions, the bound is replaced by  $\frac{11}{5}$ .

There is one last computation that needs to be done, namely, the special case<sup>10</sup>  $\lambda = (\frac{1}{2}, \dots, \frac{1}{2})_n = (0, \dots, 0)_n \boxplus \frac{1}{2}$ . The value of  $B_n(\lambda)$  is then  $\frac{n}{4}$ , and  $D^\lambda = 2^n$ . Thus, in this special case,

$$(D^\lambda)^2 e^{-t_{n,\varepsilon} B_n(\lambda)} \leq e^{n \log 4 - \frac{n \log(2n+1)}{2}} e^{-\frac{\varepsilon n \log(2n+1)}{2}} \leq \frac{11}{4} \frac{1}{(2n+1)^\varepsilon}$$

for every  $n \geq 5$ . On the other hand,

$$B_n(\lambda) = \frac{1}{2n+1} \sum_{i=1}^n \lambda_i^2 + i(2n-i)(\lambda_i - \lambda_{i+1}) \geq \frac{|\lambda|}{2n+1} + \frac{n}{2n+1} \sum_{i=1}^n i(\lambda_i - \lambda_{i+1}) = \frac{(n+1)|\lambda|}{2n+1} \geq \frac{|\lambda|}{2},$$

so we can now write:

$$\begin{aligned} S_n(t_{n,\varepsilon}) &\leq \frac{11}{4} \frac{1}{(2n+1)^\varepsilon} + \sum_{\lambda \in \mathfrak{Y}_n^*} (D^\lambda)^2 e^{-t_{n,\varepsilon} B_n(\lambda)} + (D^{\lambda \boxplus \frac{1}{2}})^2 e^{-t_{n,\varepsilon} B_n(\lambda \boxplus \frac{1}{2})} \\ &\leq \frac{11}{4} \frac{1}{(2n+1)^\varepsilon} + \sum_{\lambda \in \mathfrak{Y}_n^*} \left( (D^\lambda)^2 e^{-t_{n,0} B_n(\lambda)} + (D^{\lambda \boxplus \frac{1}{2}})^2 e^{-t_{n,0} B_n(\lambda \boxplus \frac{1}{2})} \right) e^{-2\varepsilon \log(2n+1) B_n(\lambda)} \\ &\leq \frac{11}{4} \frac{1}{(2n+1)^\varepsilon} + \sum_{\lambda \in \mathfrak{Y}_n^*} \left( \frac{121}{100} + \frac{121}{25} \right) e^{-\varepsilon |\lambda| \log(2n+1)} \\ &\leq \frac{11}{4} \frac{1}{(2n+1)^\varepsilon} + \frac{121}{20} \sum_{\lambda \in \mathfrak{Y}_n^*} \frac{1}{(2n+1)^{\varepsilon |\lambda|}} \leq \frac{33}{(2n+1)^\varepsilon} \leq \frac{144}{(2n+1)^\varepsilon} \end{aligned}$$

if one assumes  $\frac{1}{(2n+1)^\varepsilon} \leq \frac{1}{2}$ . Thus, by Proposition 16,

$$d_{\text{TV}}^{\text{SO}(2n+1, \mathbb{R})}(\mu_{2(1+\varepsilon) \log(2n+1)}, \text{Haar}) \leq \frac{6}{(2n+1)^{\frac{\varepsilon}{2}}}.$$

and again we can now remove the assumption  $\frac{1}{(2n+1)^\varepsilon} \leq \frac{1}{2}$ . The same technique applies to odd real Grassmannians, with

$$S_n\left(\frac{t_{n,\varepsilon}}{2}\right) = \sum_{\lambda \in (2\mathfrak{Y}_q \sqcup 2\mathfrak{Y}_q \boxplus 1)^*} D^\lambda e^{-\frac{t_{n,\varepsilon}}{2} B_n(\lambda)} \leq \frac{11}{10} \sum_{\lambda \in \mathfrak{Y}_n^*} e^{-\frac{\varepsilon}{2} |\lambda| \log(2n+1)} \leq \frac{55}{10(2n+1)^{\frac{\varepsilon}{2}}} \leq \frac{16}{(2n+1)^{\frac{\varepsilon}{2}}},$$

and therefore

$$d_{\text{TV}}^{\text{Gr}(2n+1, q, \mathbb{R})}(\mu_{(1+\varepsilon) \log(2n+1)}, \text{Haar}) \leq \frac{2}{(2n+1)^{\frac{\varepsilon}{4}}}.$$

In the next paragraph, we shall prove exactly the same bounds for even special orthogonal groups and even real Grassmannians.

---

<sup>10</sup>It corresponds to the *spin representation* of the odd orthogonal group  $\text{SO}(2n+1, \mathbb{R})$ .

3.2.3. *Even special orthogonal groups and their quotients.* For even special orthogonal groups, denote  $t_{n,\varepsilon} = 2(1 + \varepsilon)\log(2n)$ . The possibility of a sign  $\pm$  for the last part  $\lambda_n$  of the partitions leads to a coefficient 2 in the series  $S_n(t)$ , and on the other hand, the case of half-partitions is reduced to the case of partitions by way of an analogue of Lemma 20. Indeed,

$$\frac{D^{\lambda \boxplus \frac{1}{2}}}{D^\lambda} e^{-\frac{t_{n,0}}{2}(B_n(\lambda \boxplus \frac{1}{2}) - B_n(\lambda))} \leq e^{n \log 2 - \frac{(2n-1)\log(2n)}{8}} \leq \frac{12}{5}$$

for any  $n \geq 5$  and any partition. That said, let us analyze as before the various quotients  $\rho_{k,l}$  and  $\eta_{k,l}$  corresponding to the growth of partition described by Equation (3.2):

- $\eta_{k,n}$ : the general formula is

$$\eta_{k,n} = \left( \prod_{i=1}^{n-1} \frac{2k+2n-2i-1}{2k+n-i-1} \frac{2k+2n-2i-2}{2k+n-i-2} \right) e^{-\frac{2k+n-2}{2} \log(2n)},$$

which is decreasing in  $k$  and reduces to  $\binom{2n-1}{n} e^{-\frac{n \log(2n)}{2}}$  when  $k = 1$ . This latter bound is always smaller than 1.

- $\eta_{k,l \in \llbracket 2, n-1 \rrbracket}$ : the quotient of dimensions  $\rho_{k,l} = \rho_{k,l,(1)} \rho_{k,l,(2)}$  is equal to

$$\prod_{j=l+1}^n \frac{k+j-1+\lambda_{l+1}-\lambda_j}{k+j-l-1+\lambda_{l+1}-\lambda_j} \frac{k+\lambda_{l+1}+\lambda_j+2n-1-j}{k+\lambda_{l+1}+\lambda_j+2n-1-j-l} \prod_{1 \leq i < j \leq l} \frac{2k+2\lambda_{l+1}+2n-i-j}{2k+2\lambda_{l+1}+2n-2-i-j}.$$

The main difference with the previous computations is that  $\rho_{k,l,(2)}$  is a product over distinct indices  $i < j$ , so we will not have to worry about diagonal terms in the corresponding sum (see the argument at the beginning of §3.2.1). Hence, with the same notations as before<sup>11</sup>,

$$\begin{aligned} \log \tilde{\eta}_{k,l} &\leq -\frac{l(2k'-2+2n-l)}{2n} \log(2n) + (2k'+2n-3) \log(2k'+2n-3) \\ &\quad + (2k'+2n-2l-3) \log(2k'+2n-2l-3) - 2(2k'+2n-l-3) \log(2k'+2n-l-3) \\ &\leq -\frac{l(2v+2n-l)}{2n} \log(2n) + (2v+2n-1) \log(2v+2n-1) \\ &\quad + (2v+2n-2l-1) \log(2n-2l-1) - 2(2v+2n-l-1) \log(2v+2n-l-1); \\ \log \rho_{k,l,(1)} &\leq (2v+2n-l-1) \log(2v+2n-l-1) + (2v+n-l-1) \log(2v+n-l-1) \\ &\quad - (2v+n-1) \log(2v+n-1) - (2v+2n-2l-1) \log(2v+2n-2l-1) \\ &\quad + n \log n - l \log l - (n-l) \log(n-l) + \frac{1}{2n-2}. \end{aligned}$$

Adding together these bounds, using the concavity of  $x \log x$  and then the decreasing character with respect to  $v$  gives

$$\log \eta_{k,l} = \log \tilde{\eta}_{k,l} + \log \rho_{k,l,(1)} \leq -\frac{l(2n-l)}{2n} \log(2n) + n \log n - l \log l - (n-l) \log(n-l) + \frac{1}{2n-2},$$

which is decreasing in  $l \geq 2$ . Then,

$$-\frac{2n-2}{n} \log(2n) + n \log(n) - 2 \log 2 - (n-2) \log(n-2) + \frac{1}{2n-2}$$

is decreasing in  $n$ , and one can check that it is negative when  $n = 5$ . So,  $\eta_{k,l} \leq 1$  for any  $k$  and any  $l \in \llbracket 2, n-1 \rrbracket$ .

<sup>11</sup>We make the convention that  $0 \log 0 = 0$ , since  $2v+n-l-1$  might be equal to 0 in the following (when  $v = \lambda_{l+1} = 0$  and  $l = n-1$ ).



- $\eta_{k,1}$ : one has  $\rho_{k,1} \leq \frac{k+2n-3}{k} \frac{k+n-1}{k+n-2}$ , and therefore

$$\eta_{k,1} \leq \frac{k+2n-3}{k} \frac{k+n-1}{k+n-2} e^{-\frac{2n+2k-3}{2n} \log(2n)}.$$

Suppose  $k \geq 2$ ; then the right-hand side is smaller than  $\frac{2n-1}{2n} \frac{n+1}{2n}$ , so  $\eta_{k,1} \leq 1$ . On the other hand, for  $k = 1$ , which happens only once,

$$\eta_{1,1} \leq e^{\frac{\log(2n)}{2n}} \leq e^{\frac{\log 10}{10}} \leq \frac{4}{3}.$$

Thus, we have shown that:

**Proposition 22.** *In the case of even special orthogonal groups, at cut-off time,*

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} \leq \frac{4}{3} \quad (\text{respectively, } \frac{48}{15})$$

for any integer partition (resp. any half-partition)  $\lambda$  of length  $n$ .

Besides, the same proof as in the case of odd special orthogonal groups shows that  $B_n(\lambda) \geq \frac{|\lambda|}{2}$  for any partition. For the special half-partition  $\lambda = (0, \dots, 0)_n \boxplus \frac{1}{2}$  that cannot be treated by combining Lemmas 18 and 20, one has  $D^\lambda = 2^{n-1}$  and  $B_n(\lambda) = \frac{n}{4}$ , hence

$$(D^\lambda)^2 e^{-t_{n,\varepsilon} B_n(\lambda)} \leq e^{(n-1) \log 4 - \frac{n \log(2n)}{2}} e^{-\frac{\varepsilon n \log(2n)}{2}} \leq \frac{1}{(2n)^\varepsilon}$$

for  $n \geq 5$ . We conclude that

$$\begin{aligned} \frac{1}{2} S_n(t_{n,\varepsilon}) &\leq \frac{1}{(2n)^\varepsilon} + \sum_{\lambda \in \mathfrak{Y}_n^*} (D^\lambda)^2 e^{-t_{n,\varepsilon} B_n(\lambda)} + (D^{\lambda \boxplus \frac{1}{2}})^2 e^{-t_{n,\varepsilon} B_n(\lambda \boxplus \frac{1}{2})} \\ &\leq \frac{1}{(2n)^\varepsilon} + \sum_{\lambda \in \mathfrak{Y}_n^*} \left( \frac{16}{9} + \frac{2304}{225} \right) e^{-\varepsilon |\lambda| \log(2n)} \leq \frac{2749}{45(2n)^\varepsilon} \leq \frac{72}{(2n)^\varepsilon}, \end{aligned}$$

and therefore, by Proposition 16,

$$d_{\text{TV}}^{\text{SO}(2n, \mathbb{R})}(\mu_{2(1+\varepsilon) \log(2n)}, \text{Haar}) \leq \frac{6}{(2n)^{\frac{\varepsilon}{2}}}.$$

For even real Grassmannian varieties,

$$S_n\left(\frac{t_{n,\varepsilon}}{2}\right) = \sum_{\lambda \in (2\mathfrak{Y}_q \sqcup 2\mathfrak{Y}_q \boxplus 1)^*} D^\lambda e^{-\frac{t_{n,\varepsilon}}{2} B_n(\lambda)} \leq \frac{4}{3} \sum_{\lambda \in \mathfrak{Y}_n^*} e^{-\frac{\varepsilon}{2} |\lambda| \log(2n)} \leq \frac{20}{3(2n)^{\frac{\varepsilon}{2}}} \leq \frac{16}{(2n)^{\frac{\varepsilon}{2}}},$$

and again, the total variation distance is bounded by  $2/(2n)^{\frac{\varepsilon}{4}}$ . So, the inequalities take the same form for even and odd special orthogonal groups or real Grassmannians, and the proof of the upper bound in this case is done. The same inequality holds also for the spaces of structures  $\text{SO}(2n)/\text{U}(n)$ .

**3.2.4. Special unitary groups and their quotients.** Set  $t_{n,\varepsilon} = 2(1+\varepsilon) \log n$ . For special unitary groups, Weyl's dimension formula fortunately takes a much simpler form than before, but on the other hand, the computations on  $B_n(\lambda)$  are this time a little more subtle. We shall still prove that almost every quotient  $\eta_{k,l}$  of the quantities  $D^\lambda e^{-t_{n,0} B_n(\lambda)}$  with  $\lambda$  going from

$$(\lambda_{l+1} + k - 1, \dots, \lambda_{l+1} + k - 1, \lambda_{l+1}, \dots, \lambda_{n-1})_{n-1} \quad \text{to} \quad (\lambda_{l+1} + k, \dots, \lambda_{l+1} + k, \lambda_{l+1}, \dots, \lambda_{n-1})_{n-1}$$

is smaller than 1; but in practice, what will happen is that the negative exponentials may be much larger than before, whereas the quotients of dimensions  $\rho_{k,l}$  will be much smaller. Consider for a start  $\eta_{k,n-1}$ . One has

$$\rho_{k,n-1} = \prod_{i=1}^{n-1} \frac{k+n-i}{k-1+n-i} = \frac{k+n-1}{k},$$

whereas  $B_n(\lambda)$  is changed by  $\frac{(n-1)(n+2k-1)}{n^2}$ . So,

$$\eta_{k,n-1} = \frac{k+n-1}{k} e^{-\frac{(n-1)(n+2k-1)}{n^2} \log n} \leq \begin{cases} n e^{-\frac{n^2-1}{n^2} \log n} = e^{\frac{\log n}{n^2}} \leq 2^{\frac{1}{4}} & \text{if } k=1, \\ \frac{n+1}{2} e^{-\frac{n^2+2n-3}{n^2} \log n} \leq \frac{n+1}{2n} \leq 1 & \text{if } k \geq 2, \end{cases}$$

by using the decreasing behavior with respect to  $k$ . Notice that  $\rho_{1,n-1}$  is indeed much smaller than before (linear in  $n$  whereas before it grew exponentially in  $n$ ), but  $B_n(\lambda)$  for  $k=1$  is almost constant instead of linear in  $n$ .

In the general case,

$$\rho_{k,l} = \prod_{j=l+1}^n \frac{k' - \lambda_j + j - 1}{k' - \lambda_j + j - l - 1} \leq \prod_{j=l+1}^n \frac{k + j - 1}{k + j - l - 1}$$

with the usual notation  $k' = k + \lambda_{l+1}$ . On the other hand, the transformation on partitions makes  $B_n(\lambda)$  change by

$$\frac{-l(n-l)(n+2k'-1) + 2l|\lambda|_{l+1,n}}{n^2},$$

where  $|\lambda|_{l+1,n}$  is the restricted size  $\sum_{j=l+1}^n \lambda_j$ . Notice now that

$$-(n-l)k' + |\lambda|_{l+1,n} = \sum_{j=l+1}^n \lambda_j - \lambda_{l+1} - k \leq \sum_{j=l+1}^n -k = -(n-l)k.$$

So,

$$\eta_{k,l} \leq \prod_{j=l+1}^n \frac{k+j-1}{k+j-l-1} e^{-\frac{l(n-l)(n+2k-1)}{n^2} \log n} \leq \binom{n}{l} e^{-\frac{l(n-l)(n+1)}{n^2} \log n}$$

which can as usual be estimated by Stirling (this is the same kind of computations as before). Hence, with  $l \geq 3$ , the last bound is always smaller than 1, and also if  $l=2$  unless  $n=4$ . If  $n=4$  and  $l=2$ , then

$$\eta_{k,2} \leq \frac{(k+2)(k+3)}{k(k+1)} e^{-\frac{3+2k}{2} \log 2} \leq \begin{cases} \frac{3}{2^{3/2}} & \text{if } k=1, \\ 1 & \text{if } k \geq 2. \end{cases}$$

Finally, when  $l=1$ , one has exactly the same bound as for  $l=n-1$ , so  $2^{\frac{1}{4}}$  when  $k=1$  and 1 for  $k=2$ . Multiplying together all the bounds ( $3/2^{\frac{3}{2}}$  and twice  $2^{\frac{1}{4}}$ ), we obtain:

**Proposition 23.** *In the case of special unitary groups, at cut-off time,*

$$D^\lambda e^{-\frac{t_{n,0}}{2} B_n(\lambda)} \leq \frac{3}{2}$$

for any integer partition  $\lambda$  of length  $n-1$ .

An other big difference with the previous cases is that one cannot use Lemma 18 anymore. Indeed, for  $\lambda = (k, \dots, k)_{n-1}$ ,  $B_n(\lambda) = \frac{k(n-1)}{n} = \frac{|\lambda|}{n}$ , so there is no hope to have an inequality of the type  $B_n(\lambda) \geq k|\lambda|$  for any partition. That said, set  $\delta_i = \lambda_i - \lambda_{i+1}$ ; then,

$$B_n(\lambda) = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2 + \frac{1}{n} \sum_{i=1}^{n-1} i(n-i) \delta_i \geq \sum_{i=1}^{n-1} \frac{i(n-i)}{n} \delta_i.$$

This leads us to study the series

$$T_n(x) = \sum_{\delta_1, \dots, \delta_{n-1} \geq 0} x^{\sum_{i=1}^{n-1} \frac{i(n-i)}{n} \delta_i} = \prod_{i=1}^{n-1} \frac{1}{1 - x^{\frac{i(n-i)}{n}}}.$$

Clearly, each  $T_n(x)$  is convex on  $\mathbb{R}_+$ , so if we can show for example that  $T_n(\frac{1}{8})$  stays smaller than  $1 + \frac{K}{8}$  for every  $n$ , then we will also have the inequality  $T_n(x) \leq 1 + Kx$  for every  $0 \leq x \leq \frac{1}{8}$ . Set  $U_n(x) = \log(T_n(x))$ ; one has

$$U_n(x) = \sum_{i=1}^{n-1} -\log\left(1 - x^{\frac{i(n-i)}{n}}\right) \leq \sum_{i=1}^{n-1} x^{\frac{i(n-i)}{n}} \leq 2 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} x^{\frac{i}{2}} \leq \frac{2}{1 - x^{\frac{1}{2}}}$$

for  $0 \leq x \leq \frac{1}{8}$ . It follows that  $T_n(x) \leq 1 + Kx$  with  $K \leq 169$ . Suppose  $\frac{1}{n^{2\varepsilon}} \leq \frac{1}{8}$ . Then,

$$S_n(t_{n,\varepsilon}) = \sum_{\lambda \in \mathfrak{Y}_{n-1}^*} (D^\lambda)^2 e^{-t_{n,\varepsilon} B_n(\lambda)} \leq \frac{9}{4} \sum_{\lambda \in \mathfrak{Y}_{n-1}^*} \left(\frac{1}{n^{2\varepsilon}}\right)^{B_n(\lambda)} \leq \frac{9}{4} \left(T_n\left(\frac{1}{n^{2\varepsilon}}\right) - 1\right) \leq \frac{1521}{4n^{2\varepsilon}} \leq \frac{400}{n^{2\varepsilon}},$$

which leads to

$$d_{\text{TV}}^{\text{SU}(n,\mathbb{C})}(\mu_{2(1+\varepsilon)\log n}, \text{Haar}) \leq \frac{10}{n^\varepsilon}.$$

If  $\frac{1}{n^{2\varepsilon}} \geq \frac{1}{8}$ , then this inequality is also trivially satisfied. Hence, the case of special unitary groups is done. For the quotients  $\text{SU}(n)/\text{SO}(n)$ , one obtains

$$S_n\left(\frac{t_{n,\varepsilon}}{2}\right) \leq \frac{3}{2} \left(T_n\left(\frac{1}{n^\varepsilon}\right) - 1\right) \leq \frac{507}{2n^\varepsilon} \leq \frac{256}{n^\varepsilon}$$

and therefore

$$d_{\text{TV}}^{\text{SU}(n,\mathbb{C})/\text{SO}(n,\mathbb{R})}(\mu_{(1+\varepsilon)\log n}, \text{Haar}) \leq \frac{8}{n^{\frac{\varepsilon}{2}}}.$$

The proof is exactly the same for  $\text{SU}(2n)/\text{USp}(n)$  and gives the same inequality, however with  $(2n)^{\frac{\varepsilon}{2}}$  instead of  $n^{\frac{\varepsilon}{2}}$ .

For the complex Grassmannian varieties, we have seen that it was easier to see them as quotients of  $\text{U}(n)$  (instead of  $\text{SU}(n)$ ), and this forces us to do some additional computations. Though the cut-off phenomenon also holds in the case of  $\text{U}(n)$ , the set of irreducible representations is then labelled by sequences of possibly negative integers, which makes our scheme of growth of partitions a little bit more cumbersome to apply. Fortunately, for Grassmannians, the spherical representations can be labelled by true partitions, but then the dimensions are given by a different formula and we have to do once again the estimates of quotients  $\rho_{k,l}$  and  $\eta_{k,l}$ . For a partition of size  $p = \lfloor \frac{n}{2} \rfloor$ , one has  $B_n(\lambda) = \frac{2}{n} \sum_{i=1}^p \lambda_i^2 + (n+1-2i)\lambda_i$  and either

$$A_n(\lambda) = \left( \prod_{i=1}^p \prod_{j=1}^p \frac{\lambda_i + \lambda_j + n + 1 - i - j}{n + 1 - i - j} \right) \left( \prod_{1 \leq i < j \leq p} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right)^2$$

if  $n = 2p$ , or

$$A_n(\lambda) = \left( \prod_{i=1}^{p+1} \prod_{j=1}^{p+1} \frac{\lambda_i + \lambda_j + n + 1 - i - j}{n + 1 - i - j} \right) \left( \prod_{1 \leq i < j \leq p} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right)^2$$

when  $n = 2p + 1$ . Let us give the details when  $n = 2p$ . Again, one looks at  $\rho_{k,l} = A_n(\lambda)/A_n(\mu)$  and  $\eta_{k,l} = \rho_{k,l} e^{-\log n (B_n(\lambda) - B_n(\mu))}$ , with  $\mu$  and  $\lambda$  equal to

$$(\lambda_{l+1} + k - 1, \dots, \lambda_{l+1} + k - 1, \lambda_{l+1}, \dots, \lambda_p)_p \quad \text{and} \quad (\lambda_{l+1} + k, \dots, \lambda_{l+1} + k, \lambda_{l+1}, \dots, \lambda_p)_p.$$

The quotient of dimensions is

$$\rho_{k,l} = \left( \prod_{j=1}^l \frac{(2k' + n - j)(2k' + n - j - 1)}{(2k' + n - j - l)(2k' + n - j - l - 1)} \right) \left( \prod_{j=l+1}^p \frac{(k' - \lambda_j + j - 1)(k' + \lambda_j + n - j)}{(k' - \lambda_j + j - l - 1)(k' + \lambda_j + n - j - l)} \right)^2,$$

and a lower bound is then obtained by the usual replacement  $\lambda_{l+1} = \lambda_j = 0$  and then  $k = 1$ :

$$\rho_{k,l} \leq \frac{n-2l+1}{n+1} \binom{n+1}{l}.$$

This leads to the inequality

$$\eta_{k,l} \leq \frac{n-2l+1}{n+1} \binom{n+1}{l}^2 e^{-\frac{2l(n+1-l)}{n} \log n}$$

The last quantity is decreasing in  $l$ , as the quotient of two consecutive terms of parameters  $n, l$  and  $n, l+1$  is smaller than

$$\left( \frac{n+1-l}{l+1} e^{-\frac{n-2l}{n} \log n} \right)^2 \leq 1.$$

So,

$$\eta_{k,l} \leq \frac{n-1}{n+1} (n+1)^2 e^{-2 \log n} = \frac{n^2-1}{n^2} \leq 1.$$

It follows that  $A_n(\lambda) e^{-\log n B_n(\lambda)}$  is smaller than 1 for any partition (we leave to the reader the verification of the other case  $n = 2p+1$ , which is very similar). Finally, one can compare directly  $B_n(\lambda)$  to  $|\lambda|$ :

$$B_n(\lambda) = \frac{2}{n} \sum_{i=1}^p \lambda_i^2 + (n+1-2i)\lambda_i \geq 2 \sum_{i=1}^p \frac{i(n-i)}{n} (\lambda_i - \lambda_{i+1}) \geq \sum_{i=1}^p i(\lambda_i - \lambda_{i+1}) = |\lambda|.$$

We conclude that

$$S_n \left( \frac{t_{n,\varepsilon}}{2} \right) \leq \sum_{\lambda \in \mathfrak{Y}_q^*} e^{-\varepsilon |\lambda| \log n} \leq \frac{5}{n^\varepsilon} \leq \frac{16}{n^\varepsilon} \quad ; \quad d_{\text{TV}}^{\text{Gr}(n,q,\mathbb{C})}(\mu_{(1+\varepsilon) \log n}, \text{Haar}) \leq \frac{2}{n^{\frac{\varepsilon}{2}}}$$

and this ends the proof of all the upper bounds of type (1.11).

## 4. LOWER BOUNDS BEFORE THE CUT-OFF TIME

The proofs of the lower bounds before cut-off time rely on the following simple ideas. Denote  $\lambda_{\min}$  the (spherical) irreducible representation “of minimal eigenvalue” identified in Section 3.1. We then consider the following random variable:

$$\Omega = \begin{cases} \chi^{\lambda_{\min}}(k) & \text{in the case of groups,} \\ \sqrt{D^{\lambda_{\min}}} \phi^{\lambda_{\min}}(gK) & \text{in the case of symmetric spaces of type non-group.} \end{cases} \quad (4.1)$$

In this equation,  $k$  or  $gK$  will be taken at random either under the Haar measure of the space, or under a marginal law  $\mu_t$  of the Brownian motion; we shall denote  $\mathbb{E}_{\infty}$  and  $\mathbb{E}_t$  the corresponding expectations. When  $\Omega$  is real valued, we also denote  $\text{Var}_{\infty}$  and  $\text{Var}_t$  the corresponding variances:

$$\text{Var}[\Omega] = \mathbb{E}[\Omega^2] - \mathbb{E}[\Omega]^2 = \mathbb{E}[(\Omega - \mathbb{E}[\Omega])^2].$$

In the case of unitary groups and their quotients,  $\Omega$  will be complex valued, and we shall use the notations  $\text{Var}_{\infty}$  and  $\text{Var}_t$  for the expectation of the square of the module of  $\Omega - \mathbb{E}[\Omega]$ :

$$\text{Var}[\Omega] = \mathbb{E}[|\Omega|^2] - |\mathbb{E}[\Omega]|^2 = \mathbb{E}[|\Omega - \mathbb{E}[\Omega]|^2].$$

The normalization of Equation (4.1) is actually chosen so that  $\Omega$  is in any case of mean 0 and variance 1 under the Haar measure.

*Remark.* In fact, much more is known about the asymptotic distribution of these functions under Haar measure, when  $n$  goes to infinity; see [DS94]. For instance, over the unitary groups, the moments of order smaller than  $n_0$  of  $\chi^{(1,0,\dots,0)}(g) = \text{tr } g$  agree with those of a standard complex gaussian variable as soon as  $n$  is bigger than  $n_0$ . In particular, if  $g$  is distributed according to the Haar measure of  $U(n, \mathbb{C})$ , then  $\text{tr } g$  converges (without any normalization) towards a standard complex gaussian variable. One has similar results for orthogonal and symplectic groups, this time with standard real gaussian variables.

One will also prove that under a marginal law  $\mu_t$ , the variance of  $\Omega$  stays small for every value of  $t$ , whereas its mean before cut-off time is big (not at all near zero). Standard methods of moments allow then to prove that the probability of a event

$$E_{\alpha} = \{k \mid |\Omega(k)| \geq \alpha\} \quad \text{or} \quad \{gK \mid |\Omega(gK)| \geq \alpha\}$$

is before cut-off time near 1 under  $\mu_t$ , and near 0 under Haar measure (for an adequate choice of  $\alpha$ ). This is sufficient to prove the lower bounds, see §4.2; in other words,  $\Omega$  is a discriminating random variable for the cut-off phenomenon.

The method presented above reduces the problem mainly to the expansion in irreducible characters or in spherical zonal functions of  $\Omega^2$  or of  $|\Omega|^2$ ; cf. §4.1. In the case of compact groups, this amounts simply to understand the tensor product of  $V^{\lambda_{\min}}$  with itself, or with its conjugate when the character  $\Omega$  is complex valued. However, for compact symmetric spaces of type non-group, this is far less obvious. Notice that a zonal spherical function  $\phi^{\lambda}$  of a space  $G/K$  can be uniquely characterized by the following properties:

- it is a linear combination of matrix coefficients of the representation  $V^{\lambda}$ :

$$\phi^{\lambda}(gK) = \sum_{i=1}^{D^{\lambda}} \sum_{j=1}^{D^{\lambda}} c^{ij} \rho_{ij}^{\lambda}(gK).$$

- it is in  $\mathcal{L}^2(G/K)^K$ , i.e., it is  $K$ -bi-invariant; and it is normalized so that  $\phi^{\lambda}(eK) = 1$ .

Consequently, if  $(V^{\lambda_{\min}})^{\otimes 2} = V^{\nu_1} \oplus \dots \oplus V^{\nu_s} \oplus V^{\epsilon_1} \oplus \dots \oplus V^{\epsilon_t}$  with the  $V^{\nu_i}$  spherical irreducible representations and the  $V^{\epsilon_j}$  non-spherical irreducible representations, then there exists an expansion

$$(\phi^{\lambda_{\min}})^2 = c_{\nu_1} \phi^{\nu_1} + c_{\nu_2} \phi^{\nu_2} + \dots + c_{\nu_s} \phi^{\nu_s}. \quad (4.2)$$

Nonetheless, it seems difficult to guess at the same time the values of the coefficients  $c_\nu$  in this expansion<sup>12</sup>.

*Example.* Let us examine in detail the case of the complex projective line

$$\mathbb{P}^1(\mathbb{C}) = \mathrm{SU}(2, \mathbb{C}) / \mathrm{S}(\mathrm{U}(1, \mathbb{C}) \times \mathrm{U}(1, \mathbb{C})) = \mathrm{U}(2, \mathbb{C}) / (\mathrm{U}(1, \mathbb{C}) \times \mathrm{U}(1, \mathbb{C})).$$

As we shall see later, the label  $\lambda_{\min} = (1, 0, \dots, 0, -1)_n$  of the discriminating representation in the case of a complex Grassmannian varieties  $\mathrm{Gr}(n, q, \mathbb{C}) = \mathrm{U}(n, \mathbb{C}) / (\mathrm{U}(p, \mathbb{C}) \times \mathrm{U}(q, \mathbb{C}))$  corresponds to the adjoint representation of  $\mathrm{SU}(n, \mathbb{C})$  on  $\mathfrak{sl}(n, \mathbb{C})$ . In the following, we use the traditional embedding of  $\mathrm{U}(p, \mathbb{C}) \times \mathrm{U}(q, \mathbb{C})$  into  $\mathrm{U}(n, \mathbb{C})$  by block diagonal matrices of sizes  $p = n - q$  and  $q$ . A scalar product on  $\mathfrak{sl}(n, \mathbb{C})$  for which  $\mathrm{SU}(n, \mathbb{C})$  acts by isometry is  $\langle M | N \rangle = \mathrm{tr} MN^\dagger$ ; and a spherical vector for the subgroup  $\mathrm{S}(\mathrm{U}(p, \mathbb{C}) \times \mathrm{U}(q, \mathbb{C}))$  is

$$M_{p,q} = e^{(1,0,\dots,0,-1)_n} = \frac{1}{\sqrt{npq}} \begin{pmatrix} -q I_p & 0 \\ 0 & p I_q \end{pmatrix}.$$

As a consequence, if  $(g_{ij})_{1 \leq i,j \leq n}$  are the coefficients of a matrix  $g \in \mathrm{SU}(n, \mathbb{C})$ , then the spherical function  $\phi^{(1,0,\dots,0,-1)_n}$  writes as

$$\begin{aligned} \phi^{(1,0,\dots,0,-1)_n}(g) &= \mathrm{tr}(M_{p,q}(gM_{p,q}g^{-1})^\dagger) = \mathrm{tr}(M_{p,q}gM_{p,q}g^{-1}) \\ &= \frac{q}{np} \left( \sum_{i=1}^p \sum_{j=1}^p |g_{ij}|^2 \right) + \frac{p}{nq} \left( \sum_{i=p+1}^n \sum_{j=p+1}^n |g_{ij}|^2 \right) - \frac{1}{n} \left( \sum_{i=1}^p \sum_{j=p+1}^n |g_{ij}|^2 + \sum_{i=p+1}^n \sum_{j=1}^p |g_{ij}|^2 \right) \\ &= \frac{1}{p} \left( \sum_{i=1}^p \sum_{j=1}^p |g_{ij}|^2 \right) + \frac{1}{q} \left( \sum_{i=p+1}^n \sum_{j=p+1}^n |g_{ij}|^2 \right) - 1 \end{aligned} \quad (4.3)$$

by using on the third line the fact that rows and columns of a unitary matrix are of norm 1. In particular, the random variable  $\Omega$  is real-valued. Now, it can be shown (by a calculation with Schur functions) that

$$\begin{aligned} (V^{(1,0,\dots,0,-1)_n})^{\otimes 2} &= V^{(0,\dots,0)_n} \oplus V^{(2,0,\dots,0,-2)_n} \oplus \mathbf{1}_{n \geq 3} V^{(1,0,\dots,0,-1)_n} \oplus \mathbf{1}_{n \geq 4} V^{(1,1,0,\dots,0,-1,-1)_n} \\ &\quad \oplus V^{(1,0,\dots,0,-1)_n} \oplus \mathbf{1}_{n \geq 3} V^{(2,0,\dots,0,-1,-1)_n} \oplus \mathbf{1}_{n \geq 3} V^{(1,1,0,\dots,0,-2)_n}. \end{aligned}$$

Actually, the first line is the decomposition in irreducibles of the symmetric square  $\mathcal{S}^2(\mathfrak{sl}(n, \mathbb{C}))$ , whereas the second line is the decomposition of the skew-symmetric square  $\mathcal{A}^2(\mathfrak{sl}(n, \mathbb{C}))$ . The two last irreducible representations are not spherical, so there should be an expansion of the form

$$(\phi^{(1,0,\dots,0,-1)_n})^2 = \frac{1}{n^2 - 1} + a \phi^{(1,0,\dots,0,-1)_n} + b \phi^{(2,0,\dots,0,-2)_n} + c \phi^{(1,1,0,\dots,0,-1,-1)_n}.$$

But since one does not know *a priori* what are the spherical functions  $\phi^{(2,0,\dots,0,-2)_n}$  and  $\phi^{(1,1,0,\dots,0,-1,-1)_n}$ , it seems really hard to find by this direct algebraic approach the coefficients  $a, b, c$ .

In the case of  $\mathbb{P}^1(\mathbb{C})$ , one can proceed as follows. The matrices of  $\mathrm{SU}(2, \mathbb{C})$  write uniquely as  $\begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix}$  with  $|w|^2 + |z|^2 = 1$ , and the zonal spherical function  $\phi^{1,-1}$  is then

$$\phi^{1,-1}(w, z) = 2|w|^2 - 1.$$

Denote  $H = E_{11} - E_{22}$ ,  $X = E_{12}$  and  $Y = E_{21}$  the generators of the complex Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ ; the decomposition  $(\mathfrak{sl}(2, \mathbb{C}))^{\otimes 2} = V^{0,0} \oplus V^{2,-2} \oplus V^{1,-1}$  can be seen as a decomposition in irreducible  $\mathfrak{sl}(2, \mathbb{C})$ -modules. The first space is generated by the Casimir element

$$C = 2 \sum_{i,j=1}^2 E_{ij} \otimes E_{ji} - \sum_{i,j=1}^2 E_{ii} \otimes E_{jj}$$

<sup>12</sup>The only “easy” computation is the coefficient of the constant function in  $(\phi^\lambda)^2$ , or more generally in a product  $\phi^\lambda \phi^\rho$ . Indeed,

$$c_{\phi^1 \phi^1}[\phi^\lambda \phi^\rho] = \int_X \phi^\lambda(x) \phi^\rho(x) dx = \begin{cases} 0 & \text{if } \phi^\rho \neq \overline{\phi^\lambda}, \\ \frac{1}{D^\lambda} & \text{otherwise.} \end{cases}$$

and it corresponds to the constant function. The second space is generated by the five symmetric tensors

$$\begin{aligned} S_a &= 3\mathcal{S}(X, Y) - C \quad ; \quad S_b = \mathcal{S}(H, X) \quad ; \quad S_c = \mathcal{S}(H, Y) \quad ; \\ S_d &= \mathcal{S}(X, X) \quad ; \quad S_e = \mathcal{S}(Y, Y). \end{aligned}$$

where  $\mathcal{S}(V, W) = V \otimes W + W \otimes V$ . The action of  $H$  reads then as follows:

$$H \cdot S_a = 0 \quad ; \quad H \cdot S_b = 2S_b \quad ; \quad H \cdot S_c = -2S_c \quad ; \quad H \cdot S_d = 4S_d \quad ; \quad H \cdot S_e = -4S_e.$$

As  $H$  generates linearly the subalgebra  $\mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \times \mathfrak{gl}(1, \mathbb{C}))$ , it follows that the spherical vector associated to  $V^{2,-2}$  is up to a scalar constant

$$S_a = E_{11} \otimes E_{22} + E_{22} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} - E_{11} \otimes E_{11} - E_{22} \otimes E_{22} = \mathcal{S}(X, Y) - \frac{\mathcal{S}(H, H)}{2}$$

and therefore,  $\phi^{2,-2}(w, z) = 1 - 6|wz|^2 = 1 - 6|w|^2 + 6|w|^4$ . So, one sees that

$$(\phi^{1,-1})^2 = \frac{1}{3}\phi^{0,0} + \frac{2}{3}\phi^{2,-2},$$

and in particular the spherical function  $\phi^{1,-1}$  does not appear in the right-hand side. Now, it seems quite hard to generalize this argument to the general case of  $\mathfrak{sl}(n, \mathbb{C})$  acting on  $(\mathfrak{sl}(n, \mathbb{C}))^{\otimes 2}$ . Indeed, one would first need a description of each irreducible submodule<sup>13</sup> inside  $(\mathfrak{sl}(n, \mathbb{C}))^{\otimes 2}$ , and then to find adequate  $\mathfrak{s}(\mathfrak{gl}(p, \mathbb{C}) \otimes \mathfrak{gl}(q, \mathbb{C}))$ -spherical vectors in these spaces — it seems to us the only direct algebraic way to determine the spherical functions  $\phi^{(2,0,\dots,0,-2)_n}$  and  $\phi^{(1,1,0,\dots,0,-1,-1)_n}$ .

The previous example shows how difficult it can be to find the expansion (4.2); as far as we know, for a general zonal spherical function, there is a definitive answer only in the case of symmetric spaces of rank 1, see [Gas70]. For our problem, one can fortunately give in every case a geometric description of the discriminating spherical representation and of the corresponding spherical vector. This yields an expression of  $\phi^{\lambda_{\min}}(gK)$  as a polynomial of the matrix coefficients of  $g$ , similar to Equation (4.3). Now it turns out that the joint moments of these coefficients under  $\mu_t$  and  $\mu_\infty = \text{Haar}$  can be calculated by mean of the stochastic differential equations defining the  $G$ -valued Brownian motion; see Lemma 27, which we reproduce from [Lév11, Proposition 1.4]. As  $(\phi^{\lambda_{\min}}(gK))^2$  or  $|\phi^{\lambda_{\min}}(gK)|^2$  is also a polynomial in the coefficients  $g_{ij}$ , one can therefore compute its expectation under  $\mu_t$ , and this actually gives back the coefficients in the expansion (4.2). Thus, the algebraic difficulties raised in our proof of the lower bounds will be solved by arguments of stochastic analysis.

<sup>13</sup>Even the identification of the irreducible component  $V^{(1,0,\dots,0,-1)_n}$  inside  $\mathcal{S}^2(\mathfrak{sl}(n, \mathbb{C}))$  and  $\mathcal{A}^2(\mathfrak{sl}(n, \mathbb{C}))$  is not easy. Namely, one can show that the first space is linearly generated by the symmetric tensors

$$\begin{aligned} SH_k &= \sum_{i=1}^n \mathcal{S}(E_{ik}, E_{ki}) - \mathcal{S}(E_{i(k+1)}, E_{(k+1)i}) - \frac{2}{n} \sum_{i=1}^n \mathcal{S}(E_{kk} - E_{(k+1)(k+1)}, E_{ii}); \\ SE_{kl} &= \sum_{i=1}^n \mathcal{S}(E_{il}, E_{ki}) - \frac{2}{n} \sum_{i=1}^n \mathcal{S}(E_{kl}, E_{ii}), \end{aligned}$$

whereas the second space is generated by the skew-symmetric tensors

$$AH_k = \sum_{i=1}^n \mathcal{A}(E_{ik}, E_{ki}) - \mathcal{A}(E_{i(k+1)}, E_{(k+1)i}) \quad ; \quad AE_{kl} = \sum_{i=1}^n \mathcal{A}(E_{il}, E_{ki}),$$

with  $\mathcal{A}(V, W) = V \otimes W - W \otimes V$ . Thus, the spherical vectors of label  $(2, 0, \dots, 0, -2)_n$  and  $(1, 1, 0, \dots, 0, -1, -1)_n$  lie in the orthogonal in  $\mathcal{S}^2(\mathfrak{sl}(n, \mathbb{C}))$  of the subspace generated by the  $SH_k$ 's, the  $SE_{kl}$ 's, and the Casimir element; but this does not really help us to determine these vectors.

**4.1. Expansion of the square of the discriminating zonal spherical functions.** The orthogonality of characters or of zonal spherical functions ensures that for every non-trivial (spherical) irreducible representation  $\lambda$ ,

$$\begin{aligned}\mathbb{E}_\infty[\chi^\lambda] &= \mathbb{E}_\infty[\chi^\lambda(k) \chi^{\mathbf{1}_K}(k)] = \langle \chi^\lambda \mid \chi^{\mathbf{1}_K} \rangle_{\mathcal{L}^2(K)} = 0; \\ \mathbb{E}_\infty[\sqrt{D^\lambda} \phi^\lambda] &= \sqrt{D^\lambda} \mathbb{E}_\infty[\phi^\lambda(gK) \phi^{\mathbf{1}_G}(gK)] = \sqrt{D^\lambda} \langle \phi^\lambda \mid \phi^{\mathbf{1}_K} \rangle_{\mathcal{L}^2(G/K)} = 0.\end{aligned}$$

On the other hand, the function corresponding to the trivial representation, which is just the constant function equal to 1, has of course mean 1 under the Haar measure, and also under  $\mu_t$ . On the other hand, Theorem 15 allows one to compute the mean of a non-trivial irreducible character of zonal spherical function under  $\mu_t$ :

$$\begin{aligned}\mathbb{E}_t[\chi^\lambda] &= \int_K p_t^K(k) \chi^\lambda(k) dk = [\chi^\lambda](p_t^K) = D^\lambda e^{-\frac{t}{2} B_n(\lambda)} = \left\{ A_n(\lambda) e^{-t B_n(\lambda)} \right\}^{\frac{1}{2}} \\ \mathbb{E}_t[\sqrt{D^\lambda} \phi^\lambda] &= \sqrt{D^\lambda} \int_{X=G/K} p_t^X(x) \phi^\lambda(x) dx = \sqrt{D^\lambda} \frac{[\phi^\lambda](p_t^X)}{D^\lambda} = \left\{ A_n(\lambda) e^{-t B_n(\lambda)} \right\}^{\frac{1}{2}}\end{aligned}$$

with the notations of Proposition 16, and where  $[\chi^\lambda](f)$  or  $[\phi^\lambda](f)$  denotes the coefficient of  $\chi^\lambda$  or  $\phi^\lambda$  in the expansion of  $f$ . In particular, the table of Lemma 17 yields immediately the value of  $\mathbb{E}_t[\Omega]$  in each case.

In order to estimate  $\text{Var}_t[\Omega]$  and  $\text{Var}_\infty[\Omega]$ , we now need to find a representation-theoretic interpretation of either  $\Omega^2$  when  $\Omega$  is real-valued, or of  $|\Omega|^2$  when  $\Omega$  is complex-valued. Let us begin with the case of compact groups:

**Lemma 24.** *Suppose  $G = \text{SO}(2n, \mathbb{R})$  or  $\text{SO}(2n+1, \mathbb{R})$  or  $\text{USp}(n, \mathbb{H})$ . Then  $\Omega = \chi^{(1,0,\dots,0)_n}$  is real-valued, and*

$$\Omega^2 = (\chi^{(1,0,\dots,0)_n})^2 = \chi^{(2,0,\dots,0)_n} + \chi^{(1,1,0,\dots,0)_n} + \chi^{(0,0,\dots,0)_n}. \quad (4.4)$$

*On the other hand, when  $G = \text{SU}(n, \mathbb{C})$ ,  $\Omega$  is complex-valued, and*

$$|\Omega|^2 = \chi^{(1,0,\dots,0)_{n-1}} \chi^{(1,\dots,1)_{n-1}} = \chi^{(2,1,\dots,1)_{n-1}} + \chi^{(0,0,\dots,0)_{n-1}}. \quad (4.5)$$

*Proof.* In each case,  $\Omega(k) = \text{tr } k$ , up to the map (1.2) in the symplectic case; this explains why  $\Omega$  is real-valued in the orthogonal and symplectic case, and complex-valued in the unitary case. Then, the simplest way to prove the identities (4.4) and (4.5) is by manipulating the Schur functions of type A, B, C and D; indeed, these polynomials evaluated on the eigenvalues are known to be the irreducible characters of the corresponding groups, see §2.3. We start with the special orthogonal groups. In type  $B_n$ ,  $(z_1 + \dots + z_n + z_1^{-1} + \dots + z_n^{-1} + 1)^2$  is indeed equal to the sum of the three terms

$$\begin{aligned}sb_{(2,0,\dots,0)}(Z, Z^{-1}, 1) &= \left( \sum_{1 \leq i < j \leq n} z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1} \right) + \left( \sum_{i=1}^n z_i + z_i^{-1} \right) - n; \\ sb_{(1,1,0,\dots,0)}(Z, Z^{-1}, 1) &= \left( \sum_{1 \leq i < j \leq n} z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1} \right) + \left( \sum_{i=1}^n z_i + z_i^{-1} \right) + n; \\ sb_{(0,\dots,0)}(Z, Z^{-1}, 1) &= 1;\end{aligned}$$



whereas in type  $D_n$ ,  $(z_1 + \cdots + z_n + z_1^{-1} + \cdots + z_n^{-1})^2$  is equal to the sum of the three terms

$$\begin{aligned} sd_{(2,0,\dots,0)}(Z, Z^{-1}) &= \left( \sum_{1 \leq i \leq j \leq n} z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1} \right) - n - 1; \\ sd_{(1,1,0,\dots,0)}(Z, Z^{-1}) &= \left( \sum_{1 \leq i < j \leq n} z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1} \right) + n; \\ sd_{(0,\dots,0)}(Z, Z^{-1}) &= 1. \end{aligned}$$

For compact symplectic groups, hence in type  $C_n$ ,  $(z_1 + \cdots + z_n + z_1^{-1} + \cdots + z_n^{-1})^2$  is indeed equal to the sum of the three terms

$$\begin{aligned} sc_{(2,0,\dots,0)}(Z, Z^{-1}) &= \left( \sum_{1 \leq i \leq j \leq n} z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1} \right) - n; \\ sc_{(1,1,0,\dots,0)}(Z, Z^{-1}) &= \left( \sum_{1 \leq i < j \leq n} z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1} \right) + n - 1; \\ sc_{(0,\dots,0)}(Z, Z^{-1}) &= 1; \end{aligned}$$

and this is also  $(sc_{(1,0,\dots,0)}(Z, Z^{-1}))^2 = (\chi^{(1,0,\dots,0)}(k))^2 = \Omega(k)^2$ . Thus, Formula (4.4) is proved.

In type  $A_{n-1}$ , notice that for every character  $\chi^\lambda$ ,  $\overline{\chi^\lambda(k)} = \chi^\lambda(k^{-1}) = \chi^{\lambda^*}(k)$ , where  $\lambda^*$  is the sequence obtained from  $\lambda$  by the simple transformation

$$(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1})_{n-1} \mapsto (\lambda_1 \geq \lambda_1 - \lambda_{n-1} \geq \cdots \geq \lambda_1 - \lambda_2)_{n-1}. \quad (4.6)$$

Indeed, if  $z_1, \dots, z_n$  are the eigenvalues of  $k$ , then

$$\begin{aligned} \overline{\chi^\lambda(k)} &= s_{(\lambda_1, \dots, \lambda_{n-1})_{n-1}}(z_1^{-1}, \dots, z_n^{-1}) = s_{(\lambda_1, \dots, \lambda_{n-1}, 0)_n}(z_1^{-1}, \dots, z_n^{-1}) = s_{(0, -\lambda_{n-1}, \dots, -\lambda_1)_n}(z_n, \dots, z_1) \\ &= s_{(\lambda_1, \lambda_1 - \lambda_{n-1}, \dots, 0)_n}(z_1, \dots, z_n) = s_{(\lambda_1, \lambda_1 - \lambda_{n-1}, \dots, \lambda_1 - \lambda_2)_{n-1}}(z_1, \dots, z_n) = \chi^{\lambda^*}(k) \end{aligned}$$

Here, one uses the relation  $z_1 z_2 \cdots z_n = 1$  for every element of the torus of  $SU(n, \mathbb{C})$ , which enables one to transform a  $n$ -vector of possibly negative integers into a  $(n-1)$ -vector of non-negative integers<sup>14</sup>. In particular,

$$|\Omega(k)|^2 = |\chi^{(1,0,\dots,0)_{n-1}}(k)|^2 = \chi^{(1,0,\dots,0)_{n-1}}(k) \chi^{(1,1,\dots,1)_{n-1}}(k).$$

Then, a simple calculation with symmetric functions yields Formula (4.5):

$$\begin{aligned} \chi^{(1,0,\dots,0)_{n-1}}(k) \chi^{(1,1,\dots,1)_{n-1}}(k) &= (z_1 + \cdots + z_n)(z_1^{-1} + \cdots + z_n^{-1}) \\ &= \left( n - 1 + \sum_{i < j} z_i z_j^{-1} + z_i^{-1} z_j \right) + 1 \\ &= s_{(1,0,\dots,0,-1)_n}(Z) + s_{(0,\dots,0)_n}(Z) = s_{(2,1,\dots,1)_{n-1}}(Z) + s_{(0,\dots,0)_{n-1}}(Z) \\ &= \chi^{(2,1,\dots,1)_{n-1}}(k) + \chi^{(0,\dots,0)_{n-1}}(k) \end{aligned}$$

where  $Z = \{z_1, \dots, z_n\}$  is the alphabet of the eigenvalues of  $k$ . □

<sup>14</sup>The computation is probably much clearer on the unitary group  $U(n, \mathbb{C})$ , where we do not have to do this transformation of weights.

4.1.1. *Values of the zonal functions and abstract expansions of their squares.* As explained in the introduction of this part, the case of compact symmetric spaces of type non-group is much more involved. We start by finding an expression of  $\Omega(gK)$  in terms of the matrix coefficients  $g_{ij}$  of the matrix  $g$ .

**Proposition 25.** *In terms of the matrix coefficients of  $g$ ,  $\phi^{\lambda_{\min}}(gK)$  is given by the following formulas:*

$G/K$	$V^{\lambda_{\min}}$	$\phi^{\lambda_{\min}}(gK)$	$\mathbb{k}$
$\text{Gr}(n, q, \mathbb{R})$	$\mathfrak{so}^\perp(n, \mathbb{C})$	$\frac{1}{p} \sum_{i=1}^p \sum_{j=1}^p (g_{ij})^2 + \frac{1}{q} \sum_{i=p+1}^n \sum_{j=p+1}^n (g_{ij})^2 - 1$	$\mathbb{R}$
$\text{Gr}(n, q, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C})$	$\frac{1}{p} \sum_{i=1}^p \sum_{j=1}^p  g_{ij} ^2 + \frac{1}{q} \sum_{i=p+1}^n \sum_{j=p+1}^n  g_{ij} ^2 - 1$	$\mathbb{R}$
$\text{Gr}(n, q, \mathbb{H})$	$\mathfrak{sp}^\perp(2n, \mathbb{C})$	$\frac{1}{p} \sum_{i=1}^p \sum_{j=1}^p  g_{ij} ^2 + \frac{1}{q} \sum_{i=p+1}^n \sum_{j=p+1}^n  g_{ij} ^2 - 1$	$\mathbb{R}$
$\text{SO}(2n, \mathbb{R})/\text{U}(n, \mathbb{C})$	$\mathcal{A}^2(\mathbb{C}^{2n})$	$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n g_{(2i)(2j)} g_{(2i-1)(2j-1)} - g_{(2i)(2j-1)} g_{(2i-1)(2j)}$	$\mathbb{R}$
$\text{SU}(n, \mathbb{C})/\text{SO}(n, \mathbb{R})$	$\mathcal{S}^2(\mathbb{C}^n)$	$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n (g_{ij})^2$	$\mathbb{C}$
$\text{SU}(2n, \mathbb{C})/\text{USp}(n, \mathbb{H})$	$\mathcal{A}^2(\mathbb{C}^{2n})$	$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n g_{(2i)(2j)} g_{(2i-1)(2j-1)} - g_{(2i)(2j-1)} g_{(2i-1)(2j)}$	$\mathbb{C}$
$\text{USp}(n, \mathbb{H})/\text{U}(n, \mathbb{C})$	$\mathcal{S}^2(\mathbb{C}^{2n})$	$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n ([i](g_{ij}))^2 + ([j](g_{ij}))^2 - ([i](g_{ij}))^2 - ([j](g_{ij}))^2$	$\mathbb{R}$

For real Grassmannians,  $\mathfrak{so}^\perp(n, \mathbb{C})$  denotes the orthogonal complement of  $\mathfrak{so}(n, \mathbb{C})$  in  $\mathfrak{sl}(n, \mathbb{C})$ ; and for quaternionic Grassmannians,  $\mathfrak{sp}^\perp(2n, \mathbb{C})$  denotes the orthogonal complement of  $\mathfrak{sp}(2n, \mathbb{C})$  in  $\mathfrak{sl}(2n, \mathbb{C})$ .

*Proof.* Each space  $V^{\lambda_{\min}}$  described in the statement of our proposition is endowed with a natural action of  $G = \text{SO}(n)$  or  $\text{SU}(n)$  or  $\text{USp}(n)$ , namely, the action by conjugation in the case of Grassmannian varieties, and the diagonal action on tensors in the case of spaces of structures. Then, to say that

$$\begin{aligned}
V_{\text{SO}(n, \mathbb{R})}^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} &= \mathfrak{so}^\perp(n, \mathbb{C}) & ; & & V_{\text{U}(n, \mathbb{C})}^{(1,0,\dots,0,-1)_n} &= \mathfrak{sl}(n, \mathbb{C}) & ; & & V_{\text{USp}(n, \mathbb{H})}^{(1,1,0,\dots,0)_n} &= \mathfrak{sp}^\perp(2n, \mathbb{C}) & ; \\
V_{\text{SO}(2n, \mathbb{R})}^{(1,1,0,\dots,0)_n} &= \mathcal{A}^2(\mathbb{C}^n) & ; & & V_{\text{SU}(n, \mathbb{C})}^{(2,0,\dots,0)_{n-1}} &= \mathcal{S}^2(\mathbb{C}^n) & ; & & V_{\text{SU}(2n, \mathbb{C})}^{(1,1,0,\dots,0)_{2n-1}} &= \mathcal{A}^2(\mathbb{C}^{2n}) & ; \\
V_{\text{USp}(n, \mathbb{H})}^{(2,0,\dots,0)_n} &= \mathcal{S}^2(\mathbb{C}^{2n})
\end{aligned}$$

is equivalent to the following statements: the trace of  $g \in \text{SO}(n, \mathbb{R})$  acting on  $\mathfrak{so}^\perp(n, \mathbb{C})$  is given by the Schur function of type B or D and label  $(2, 0, \dots, 0)_{\lfloor \frac{n}{2} \rfloor}$ ; the trace of  $g \in \text{U}(n, \mathbb{C})$  acting on  $\mathfrak{sl}(n, \mathbb{C})$  is given by the Schur function of type A and label  $(1, 0, \dots, 0, -1)_n$ ; etc. Let us detail for instance this last case. We have seen in the previous Lemma that

$$s_{(1,0,\dots,0,-1)_n}(Z) = (z_1 + \dots + z_n)(z_1^{-1} + \dots + z_n^{-1}) - 1.$$

On the other hand, the module  $\mathfrak{gl}(n, \mathbb{C})$  on which  $\text{SU}(n, \mathbb{C})$  acts by conjugation is the tensor product of modules  $(\mathbb{C}^n) \otimes (\mathbb{C}^n)^*$ . It follows that the trace of the action by conjugation of  $g \in \text{SU}(n, \mathbb{C})$  on  $\mathfrak{gl}(n, \mathbb{C})$  is

$$\chi(g) = (\text{trg}) (\text{tr}(g^{-1})^t) = (z_1 + \dots + z_n)(z_1^{-1} + \dots + z_n^{-1})$$

if  $z_1, \dots, z_n$  are the eigenvalues of  $g$ . Subtracting 1 amounts to look at the irreducible submodule  $\mathfrak{sl}(n, \mathbb{C})$  inside  $\mathfrak{gl}(n, \mathbb{C})$ . The other cases are entirely similar, and the corresponding values of the Schur functions have all been computed in Lemma 24.

Once the discriminating representations have been given a geometric interpretation, it is easy to find the corresponding  $K$ -invariant (spherical) vectors. We endow each space of matrices with the invariant scalar product  $\langle M | N \rangle = \text{tr} MN^\dagger$ , and each space of 2-tensors with the scalar product  $\langle x_1 \otimes x_2 | y_1 \otimes y_2 \rangle =$

$\langle x_1 | y_1 \rangle \langle x_2 | y_2 \rangle$ , where  $\langle v | w \rangle$  is the usual Hermitian scalar product on  $\mathbb{C}^n$  or  $\mathbb{C}^{2n}$ . We also denote  $(e_i)_i$  the canonical basis of  $\mathbb{C}^n$  or  $\mathbb{C}^{2n}$ . Then, the  $K$ -spherical vectors write as:

$G$	$K$	$e^{\lambda_{\min}}$
$\mathrm{SO}(n)$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$	$\frac{1}{\sqrt{npq}} \begin{pmatrix} -qI_p & 0 \\ 0 & pI_q \end{pmatrix}$
$\mathrm{SU}(n)$	$\mathrm{S}(U(p) \times U(q))$	$\frac{1}{\sqrt{npq}} \begin{pmatrix} -qI_p & 0 \\ 0 & pI_q \end{pmatrix}$
$\mathrm{USp}(n)$	$\mathrm{USp}(p) \times \mathrm{USp}(q)$	$\frac{1}{\sqrt{2npq}} \begin{pmatrix} -qI_{2p} & 0 \\ 0 & pI_{2q} \end{pmatrix}$
$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$\frac{1}{\sqrt{2n}} \sum_{i=1}^n e_{2i} \otimes e_{2i-1} - e_{2i-1} \otimes e_{2i}$
$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$
$\mathrm{SU}(2n)$	$\mathrm{USp}(n)$	$\frac{1}{\sqrt{2n}} \sum_{i=1}^n e_{2i} \otimes e_{2i-1} - e_{2i-1} \otimes e_{2i}$
$\mathrm{USp}(n)$	$\mathrm{U}(n)$	$\frac{1}{\sqrt{2n}} \sum_{i=1}^{2n} e_i \otimes e_i$

In each case,  $e^{\lambda_{\min}}$  belongs trivially to  $V^{\lambda_{\min}}$  and is of norm 1, so the only thing to check then is the  $K$ -invariance. In the case of Grassmannian varieties, the matrix  $e^{\lambda_{\min}}$  commutes indeed with  $G(p) \times G(q)$ , since it is also  $(p, q)$ -block-diagonal and with scalar multiples of the identity matrix in each diagonal block. For  $\mathrm{U}(n)$  inside  $\mathrm{SO}(2n)$ , we use the well-known fact that inside<sup>15</sup>  $\mathrm{SL}(2n, \mathbb{C})$ ,

$$\mathrm{SO}(2n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{C}) \simeq \mathrm{U}(n, \mathbb{C}), \quad (4.7)$$

the isomorphism being given by the map (1.1). This implies in particular that  $\mathrm{U}(n)$  leaves invariant the skew-symmetric tensor  $\sum_{i=1}^n e_{2i} \otimes e_{2i-1} - e_{2i-1} \otimes e_{2i}$  corresponding to the skew-symmetric form defining  $\mathrm{Sp}(2n, \mathbb{C})$ . The intersection formula (4.7) also proves that  $\mathrm{U}(n)$  leaves invariant the symmetric tensor  $\sum_{i=1}^{2n} e_i \otimes e_i$ , whence the value of the spherical vector for  $\mathrm{U}(n)$  inside  $\mathrm{USp}(n)$ . Finally, for  $\mathrm{SO}(n)$  inside  $\mathrm{SU}(n)$  and  $\mathrm{USp}(n)$  inside  $\mathrm{SU}(2n)$ , we use again the defining symmetric bilinear form or skew-symmetric bilinear form associated to the group  $K$  to construct a  $K$ -invariant vector. The value of  $\phi^{\lambda_{\min}}$  is then given by the formula  $\phi^{\lambda}(g) = \langle e^{\lambda} | \rho^{\lambda}(g)e^{\lambda} \rangle$ , that is to say

$$\mathrm{tr}(M_{p,q} g M_{p,q} g^{\dagger}) \quad ; \quad \mathrm{tr}(M_{p,q} g M_{p,q} g^{\dagger}) \quad ; \quad \frac{1}{2} \mathrm{tr}(\widetilde{M}_{p,q} \widetilde{g} \widetilde{M}_{p,q} \widetilde{g}^{\dagger})$$

for real, complex and quaternionic Grassmannians;

$$\frac{1}{n} \sum_{i,j=1}^n (g_{ij})^2 \quad ; \quad \frac{1}{2n} \sum_{i,j=1}^{2n} (\widetilde{g}_{ij})^2$$

for  $\mathrm{SU}(n)/\mathrm{SO}(n)$  and  $\mathrm{USp}(n)/\mathrm{U}(n)$ ; and

$$\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n g(2i)(2j)g(2i-1)(2j-1) - g(2i)(2j-1)g(2i-1)(2j)$$

for  $\mathrm{SO}(2n)/\mathrm{U}(n)$  and  $\mathrm{SU}(2n)/\mathrm{USp}(n)$ . Here by  $\widetilde{g}$  we mean the complex matrix of size  $2n \times 2n$  obtained from a quaternionic matrix of size  $n \times n$  by the map (1.2). In this last case, the computations can in fact

<sup>15</sup>Here,  $\mathrm{Sp}(2n, \mathbb{C})$  is the (non-compact) complex symplectic group, that is to say the set of complex matrices of size  $2n \times 2n$  that leaves invariant the skew-symmetric form  $\sum_{i=1}^n e_{2i-1} \wedge e_{2i}$ . The notation  $\mathrm{USp}(n, \mathbb{H})$  used in this paper was meant to avoid any confusion between  $\mathrm{Sp}(2n, \mathbb{C})$  and its compact form, the compact symplectic group.

be done inside  $M(n, \mathbb{H})$ : indeed,

$$\begin{aligned} & (\widetilde{g}_{(2i-1)(2j-1)})^2 + (\widetilde{g}_{(2i-1)(2j)})^2 + (\widetilde{g}_{(2i)(2j-1)})^2 + (\widetilde{g}_{(2i)(2j)})^2 \\ &= 2([1](g_{ij}))^2 + ([j](g_{ij}))^2 - ([i](g_{ij}))^2 - ([k](g_{ij}))^2), \end{aligned}$$

whereas  $\widetilde{M}^\star = (\widetilde{M})^\dagger$  and  $\frac{1}{2} \operatorname{tr} \widetilde{M} = \Re(\operatorname{tr} M)$ . Thus, the formulas for the discriminating spherical functions of the spaces of structures are entirely proved, whereas for Grassmannian varieties, it suffices to check that for any unitary quaternionic matrix  $N$ ,

$$\Re(\operatorname{tr} M_{p,q} N M_{p,q} N^\star) = \frac{1}{p} \sum_{i=1}^p \sum_{j=1}^p |g_{ij}|^2 + \frac{1}{q} \sum_{i=p+1}^n \sum_{j=p+1}^n |g_{ij}|^2 - 1;$$

indeed the real and complex cases are specializations of this formula. This last calculation is identical to the one done in the example before §4.1, see Formula (4.3).  $\square$

**Lemma 26.** *There exists coefficients  $a, b, c, \dots$  (different on each line, and depending on  $n$  and  $q$ ) such that the following expansions hold:*

$$\begin{aligned} \operatorname{Gr}(n, q, \mathbb{R}) : \quad & \left( \phi^{(2,0,\dots,0) \lfloor \frac{n}{2} \rfloor} \right)^2 = \frac{2}{n^2 + n - 2} + a \phi^{(2,0,\dots,0) \lfloor \frac{n}{2} \rfloor} + b \phi^{(1,1,0,\dots,0) \lfloor \frac{n}{2} \rfloor} \\ & + c \phi^{(2,2,0,\dots,0) \lfloor \frac{n}{2} \rfloor} + d \phi^{(3,1,0,\dots,0) \lfloor \frac{n}{2} \rfloor} + e \phi^{(4,0,\dots,0) \lfloor \frac{n}{2} \rfloor}; \\ \operatorname{Gr}(n, q, \mathbb{C}) : \quad & \left( \phi^{(2,1,\dots,1)_{n-1}} \right)^2 = \frac{1}{n^2 - 1} + a \phi^{(2,1,\dots,1)_{n-1}} + b \phi^{(4,2,\dots,2)_{n-1}} + c \phi^{(2,2,1,\dots,1,0)_{n-1}}; \\ \operatorname{Gr}(n, q, \mathbb{H}) : \quad & \left( \phi^{(1,1,0,\dots,0)_n} \right)^2 = \frac{1}{2n^2 - n - 1} + a \phi^{(1^2,0,\dots,0)_{n-1}} + b \phi^{(1^4,0,\dots,0)_n} + c \phi^{(2,2,0,\dots,0)_n}; \\ \operatorname{SO}(2n)/\operatorname{U}(n) : \quad & \left( \phi^{(1,1,0,\dots,0)_n} \right)^2 = \frac{1}{2n^2 - n} + a \phi^{(1^2,0,\dots,0)_n} + b \phi^{(1^4,0,\dots,0)_n} + c \phi^{(2,2,0,\dots,0)_n}; \\ \operatorname{SU}(n)/\operatorname{SO}(n) : \quad & \left| \phi^{(2,0,\dots,0)_{n-1}} \right|^2 = \frac{2}{n^2 + n} + a \phi^{(4,2,\dots,2)_{n-1}}; \\ \operatorname{SU}(2n)/\operatorname{USp}(n) : \quad & \left| \phi^{(1,1,0,\dots,0)_{2n-1}} \right|^2 = \frac{1}{2n^2 - n} + a \phi^{(2,2,1,\dots,1,0)_{2n-1}}; \\ \operatorname{USp}(n)/\operatorname{U}(n) : \quad & \left( \phi^{(2,0,\dots,0)_n} \right)^2 = \frac{1}{2n^2 + n} + a \phi^{(2,0,\dots,0)_n} + b \phi^{(2,2,0,\dots,0)_n} + c \phi^{(4,0,\dots,0)_n}. \end{aligned}$$

In these formulas, it is understood that if the label  $\lambda$  of the spherical function  $\phi^\lambda$  does not make sense for a choice of  $n$  and  $q$ , then this term does not appear in the expansion.

*Proof.* Each time, one computes the expansion in irreducible representations of  $V^{\lambda_{\min}} \otimes V^{\lambda_{\min}}$  in the case of real-valued spherical functions, and of  $V^{\lambda_{\min}} \otimes V^{\lambda_{\min}^*}$  in the case of complex-valued spherical functions, where  $\lambda \mapsto \lambda^*$  is the transformation of weights given by Equation (4.6). This expansion can be found with Schur functions; let us detail for instance the case of complex Grassmannian varieties  $\operatorname{Gr}(n, q, \mathbb{C})$ . With an alphabet of eigenvalues  $Z = \{z_1, \dots, z_n\}$  such that  $z_1 z_2 \cdots z_n = 1$ , one has

$$\begin{aligned} s_{(0,\dots,0)_{n-1}}(Z) &= 1 \\ s_{(2,1,\dots,1)_{n-1}}(Z) &= s_{(1,0,\dots,0,-1)_n}(Z) = \left( \sum_{i,j=1}^n z_i z_j^{-1} \right) - 1 \\ s_{(4,2,\dots,2)_{n-1}}(Z) &= s_{(2,0,\dots,0,-2)_n}(Z) = \left( \sum_{1 \leq i \leq j \leq n} \sum_{1 \leq k \leq l \leq n} z_i z_j z_k^{-1} z_l^{-1} \right) - \left( \sum_{i,j=1}^n z_i z_j^{-1} \right) \end{aligned}$$

$$\begin{aligned}
s_{(2,2,1,\dots,1,0)_{n-1}}(Z) &= s_{(1,1,0,\dots,0,-1,-1)_n}(Z) = \left( \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} z_i z_j z_k^{-1} z_l^{-1} \right) - \left( \sum_{i,j=1}^n z_i z_j^{-1} \right) \\
s_{(3,1,\dots,1,0)_{n-1}}(Z) &= s_{(2,0,\dots,0,-1,-1)_n}(Z) = \left( \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} z_i z_j z_k^{-1} z_l^{-1} \right) - \left( \sum_{i,j=1}^n z_i z_j^{-1} \right) + 1 \\
s_{(3,3,2,\dots,2)_{n-1}}(Z) &= s_{(1,1,0,\dots,0,-2)_n}(Z) = \left( \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} z_i z_j z_k^{-1} z_l^{-1} \right) - \left( \sum_{i,j=1}^n z_i z_j^{-1} \right) + 1.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(V^{(2,1,\dots,1)_{n-1}})^{\otimes 2} &= V^{(0,\dots,0)_{n-1}} \oplus 2 V^{(2,1,\dots,1)_{n-1}} \oplus V^{(4,2,\dots,2)_{n-1}} \oplus V^{(2,2,1,\dots,1,0)_{n-1}} \\
&\quad \oplus V^{(3,3,2,\dots,2)_{n-1}} \oplus V^{(3,1,\dots,1,0)_{n-1}},
\end{aligned}$$

because the same equality with Schur functions holds. The second line corresponds to non spherical representations, so only the terms of the first line can contribute to  $(\phi^{(2,1,\dots,1)_{n-1}})^2$ . Entirely similar calculations yield the following expansions:

- $\text{Gr}(n, q, \mathbb{R})$ :

$$\begin{aligned}
\left( V^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \right)^{\otimes 2} &= V^{(0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \oplus V^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \oplus V^{(1,1,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \\
&\quad \oplus V^{(2,2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \oplus V^{(3,1,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \oplus V^{(4,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}}
\end{aligned}$$

- $\text{Gr}(n, q, \mathbb{H})$ :

$$\begin{aligned}
\left( V^{(1,1,0,\dots,0)_n} \right)^{\otimes 2} &= V^{(0,\dots,0)_n} \oplus V^{(1,1,0,\dots,0)_n} \oplus V^{(1,1,1,1,0,\dots,0)_n} \oplus V^{(2,2,0,\dots,0)_n} \\
&\quad \oplus V^{(2,1,1,0,\dots,0)_n}.
\end{aligned}$$

Only the terms on the first line are spherical.

- $\text{SO}(2n, \mathbb{R})/\text{U}(n, \mathbb{C})$ :

$$\begin{aligned}
\left( V^{(1,1,0,\dots,0)_n} \right)^{\otimes 2} &= V^{(0,\dots,0)_n} \oplus V^{(1,1,0,\dots,0)_n} \oplus V^{(1,1,1,1,0,\dots,0)_n} \oplus V^{(2,2,0,\dots,0)_n} \\
&\quad \oplus V^{(2,0,\dots,0)_n} \oplus V^{(2,1,1,0,\dots,0)_n},
\end{aligned}$$

again with non-spherical representations gathered on the second line.

- $\text{SU}(n, \mathbb{C})/\text{SO}(n, \mathbb{R})$ :

$$V^{(2,0,\dots,0)_{n-1}} \otimes V^{(2,2,\dots,2)_{n-1}} = V^{(0,\dots,0)_{n-1}} \oplus V^{(4,2,\dots,2)_{n-1}} \oplus V^{(2,1,\dots,1)_{n-1}},$$

and the last term is not a spherical representation.

- $\text{SU}(2n, \mathbb{C})/\text{USp}(n, \mathbb{H})$ :

$$V^{(1,1,0,\dots,0)_{2n-1}} \otimes V^{(1,\dots,1,0)_{2n-1}} = V^{(0,\dots,0)_{2n-1}} \oplus V^{(2,2,1,\dots,1,0)_{2n-1}} \oplus V^{(2,1,\dots,1)_{2n-1}},$$

and again the last term is not spherical.

- $\text{USp}(n, \mathbb{H})/\text{U}(n, \mathbb{C})$ :

$$\begin{aligned}
\left( V^{(2,0,\dots,0)_n} \right)^{\otimes 2} &= V^{(0,\dots,0)_n} \oplus V^{(2,0,\dots,0)_n} \oplus V^{(2,2,\dots,2)_n} \oplus V^{(4,0,\dots,0)_n} \\
&\quad \oplus V^{(1,1,0,\dots,0)_n} \oplus V^{(3,1,0,\dots,0)_n}.
\end{aligned}$$

The terms on the second line corresponds to non-spherical representations.

As mentioned before, the coefficient of the constant function in  $|\phi^{\lambda_{\min}}|^2$  is then always equal to  $\frac{1}{D^{\lambda_{\min}}}$ .  $\square$

For the spaces  $SU(n)/SO(n)$  and  $SU(2n)/USp(n)$ , the remaining coefficient  $a$  can be found by evaluating the spherical functions at  $e_G$ . Thus,

$$\begin{aligned} SU(n)/SO(n) : \quad & \left| \phi^{(2,0,\dots,0)_{n-1}} \right|^2 = \frac{2}{n^2+n} + \frac{n^2+n-2}{n^2+n} \phi^{(4,2,\dots,2)_{n-1}}; \\ SU(2n)/USp(n) : \quad & \left| \phi^{(1,1,0,\dots,0)_{2n-1}} \right|^2 = \frac{1}{2n^2-n} + \frac{2n^2-n-1}{2n^2-n} \phi^{(2,2,1,\dots,1,0)_{2n-1}}. \end{aligned}$$

But in the other cases, the values of the spherical functions appearing in the right-hand side of the formulas of Lemma 26 are unfortunately not known *a priori*, which makes finding the coefficients  $a, b, c, \dots$  quite difficult. However, since one only needs to compute  $\mathbb{E}_t[(\phi^{\lambda_{\min}})^2]$ , and since  $\phi^{\lambda_{\min}}$  is explicit in terms of matrix coefficients, one can use the following Lemma (cf. [Lév11, Proposition 1.4]).

**Lemma 27.** *Let  $k \geq 1$  be any integer, and  $(g_t)_{t \in \mathbb{R}_+}$  be the Brownian motion on  $SO(n)$  or  $SU(n)$ . The joint moments of order  $k$  of the matrix coefficients of  $g_t$  are given by*

$$\mathbb{E}[g_t^{\otimes k}] = \exp \left( t \frac{k \alpha_{\mathfrak{g}}}{2} (I_n)^{\otimes k} + t \sum_{1 \leq i < j \leq k} \eta_{i,j}(C_{\mathfrak{g}}) \right) \quad (4.8)$$

where  $\alpha_{\mathfrak{g}}$  is the coefficient introduced on page 10;  $C_{\mathfrak{g}}$  is the Casimir operator; and  $\eta_{i,j}$  is the linear map from  $M(n, \mathbb{k})^{\otimes 2}$  to  $M(n, \mathbb{k})^{\otimes k}$  defined on simple tensors  $X \otimes Y$  by

$$X \otimes Y \mapsto (I_n)^{\otimes(i-1)} \otimes X \otimes (I_n)^{\otimes(j-i-1)} \otimes Y \otimes (I_n)^{\otimes(k-j)}.$$

In the complex case, one has also:

$$\mathbb{E}[(g_t)^{\otimes k} \otimes (\overline{g_t})^{\otimes l}] = \exp \left( t \frac{(k+l) \alpha_{\mathfrak{g}}}{2} (I_n)^{\otimes(k+l)} + t \sum_{1 \leq i < j \leq k+l} \tilde{\eta}_{i,j}(C_{\mathfrak{g}}) \right),$$

with

$$\tilde{\eta}_{i,j}(X \otimes Y) = \begin{cases} \eta_{i,j}(X \otimes Y) & \text{if } i, j \in \llbracket 1, k \rrbracket; \\ -\eta_{i,j}(X \otimes Y^t) & \text{if } i \in \llbracket 1, k \rrbracket \text{ and } j \in \llbracket k+1, k+l \rrbracket; \\ \eta_{i,j}(X^t \otimes Y^t) & \text{if } i, j \in \llbracket k+1, k+l \rrbracket. \end{cases}$$

*Proof.* In the complex case, recall the stochastic differential equation satisfied by  $g_t$ , and therefore by  $\overline{g_t}$ :

$$dg_t = g_t dB_t + \frac{\alpha_{\mathfrak{g}}}{2} g_t dt \quad ; \quad dg_t^* = -\overline{g_t} dB_t^t + \frac{\alpha_{\mathfrak{g}}}{2} \overline{g_t} dt.$$

Itô's formula yields then the following stochastic differential equation for  $(g_t)^{\otimes k} \otimes (\overline{g_t})^{\otimes l}$ :

$$\begin{aligned} d(g^{\otimes k} \otimes \overline{g}^{\otimes l})_t &= (g_t)^{\otimes k} \otimes (\overline{g_t})^{\otimes l} \left( \frac{(k+l) \alpha_{\mathfrak{g}}}{2} dt + \sum_{1 \leq i < j \leq k+l} \tilde{\eta}_{i,j}(dB_t \otimes dB_t) \right) \\ &+ (g_t)^{\otimes k} \otimes (\overline{g_t})^{\otimes l} \left( \sum_{i=1}^k (I_n)^{\otimes(i-1)} \otimes dB_t \otimes (I_n)^{\otimes(k+l-i)} - \sum_{i=k+1}^{k+l} (I_n)^{\otimes(i-1)} \otimes dB_t^t \otimes (I_n)^{\otimes(k+l-i)} \right). \end{aligned}$$

The quadratic variation of  $B_t$  is given by the Casimir operator:  $dB_t \otimes dB_t = C_{\mathfrak{g}} dt$ . Taking expectations in the formula above leads now to a differential equation for  $\mathbb{E}[(g_t)^{\otimes k} \otimes (\overline{g_t})^{\otimes l}]$ , whose solution is the exponential of matrices in the statement of this lemma. The real case is the specialization  $l = 0$  of the previous discussion, though with a different Casimir operator. In the quaternionic case, one has to be much more careful. In particular, since the quaternionic conjugate of  $pq$  is  $q^*p^*$  instead of  $p^*q^*$ , in the

previous argument the SDE for  $g_t^*$  does not take exactly the same form. A way to overcome this problem is to use the doubling map (1.2). Thus, we write an equation for  $\tilde{g}_t$  instead of  $g_t$ :

$$\mathbb{E}[(\tilde{g}_t)^{\otimes k}] = \exp\left(t \frac{k \alpha_{\mathfrak{g}}}{2} (I_{2n})^{\otimes k} + t \sum_{1 \leq i < j \leq k} \eta_{i,j}(C_{\mathfrak{g}})\right),$$

where the Casimir is now considered as an element of  $(\text{End}(\mathbb{C}^{2n}))^{\otimes 2}$ . As we shall see later, joint moments of the entries of  $g$  and  $g^*$  are combinations of the joint moments of the entries of  $\tilde{g}$ , so the previous formula will prove sufficient to solve our problem in the quaternionic case.  $\square$

It turns out that in each case important for our computations, the matrix  $\sum_{1 \leq i < j \leq k} \tilde{\eta}_{i,j}(C_{\mathfrak{g}})$  can be explicitly diagonalized, with a basis of eigenvectors that is quite tractable (to be honest, with the help of a computer). In the following, we describe the eigenvalues and eigenvectors of these matrices, and leave the reader check that they are indeed eigenvalues and eigenvectors: this is each time an immediate computation with elementary matrices, though quite tedious if  $k = 4$  or  $k + l = 4$ . In what follows, we write  $e[i_1, i_2, \dots, i_r] = e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r}$ .

**4.1.2. Quotients of orthogonal groups.** For special orthogonal groups, set  $\frac{1}{n}M_{n,k} = \sum_{1 \leq i < j \leq k} \eta_{i,j}(C_{\mathfrak{so}(n)})$ , viewed as an element of  $\text{End}((\mathbb{R}^n)^{\otimes k})$ . If  $k = 2$ , then the eigenvalues and eigenvectors of  $M_{n,2} = \sum_{1 \leq i < j \leq n} (E_{ij} - E_{ji})^{\otimes 2}$  are:

eigenvalue	multiplicity	eigenvectors
$n - 1$	1	$\sum_{i=1}^n e[i, i]$
1	$\frac{n(n-1)}{2}$	$e[i, j] - e[j, i], i < j$
-1	$\frac{(n+2)(n-1)}{2}$	$e[i, j] + e[j, i], i < j$ $e[i, i] - e[i + 1, i + 1], i \leq n - 1$

This allows to compute  $\exp(-\frac{(n-1)t}{n}) \exp(\frac{t}{n}M_{n,2})$ , which is the right-hand side of Formula (4.8) in the case of  $\text{SO}(n, \mathbb{R})$  and for  $k = 2$ . One obtains:

$$\begin{aligned} \mathbb{E}[(g_{ii})^2] &= \frac{1}{n} + \left(1 - \frac{1}{n}\right) e^{-t} \quad ; \quad \mathbb{E}[(g_{ij})^2] = \frac{1}{n} (1 - e^{-t}) \quad ; \\ \mathbb{E}[g_{ii}g_{jj}] &= \frac{1}{2} \left(e^{-t} + e^{-\frac{n-2}{n}t}\right) \quad ; \quad \mathbb{E}[g_{ij}g_{ji}] = \frac{1}{2} \left(e^{-t} - e^{-\frac{n-2}{n}t}\right) \quad ; \end{aligned}$$

and all the other mixed moments vanish (e.g.,  $\mathbb{E}[g_{ii}g_{ij}]$  or  $\mathbb{E}[g_{ij}g_{kl}]$ ). Now, if  $k = 4$ , then the eigenvalues and eigenvectors of  $M_{n,4}$  are:

eigenvalue	multiplicity	eigenvectors (not exhaustive, some repetitions)
$2n - 2$	3	$\sum_{k=1}^n \sum_{l=1}^n e[k, k, l, l], \star$
$n$	$3n(n - 1)$	$\sum_{k=1}^n e[i, j, k, k] - e[j, i, k, k], i < j, \star$
$n - 2$	$3(n + 2)(n - 1)$	$\sum_{k=1}^n e[i, j, k, k] + e[j, i, k, k], i < j, \star$ $\sum_{k=1}^n e[i, i, k, k] - e[i + 1, i + 1, k, k], i \leq n - 1, \star$
6	$\frac{n(n-1)(n-2)(n-3)}{24}$	$\sum_{\sigma \in \mathfrak{S}_4} \varepsilon(\sigma) e[i, j, k, l]^\sigma, i < j < k < l$
2	$\frac{3n(n+2)(n-1)(n-3)}{8}$	$D_1^\eta(i, j, k, l), D_2^\eta(i, j, k, l), D_3^\eta(i, j, k, l), i \neq j \neq k \neq l$
0	$\frac{n(n+1)(n+2)(n-3)}{6}$	$S_1(i, j, k, l), S_2(i, j, k, l), i \neq j \neq k \neq l$ $K_1(i, j, k, l), K_2(i, j, k, l), i \neq j \neq k \neq l$
-2	$\frac{3(n-1)(n-2)(n+1)(n+4)}{8}$	$\begin{pmatrix} e[i, j]^\otimes + e[j, k]^\otimes + e[k, i]^\otimes \\ -e[j, i]^\otimes - e[k, j]^\otimes - e[i, k]^\otimes \end{pmatrix}, i \neq j \neq k, \star$ $D_1^\theta(i, j, k, l), D_2^\theta(i, j, k, l), D_3^\theta(i, j, k, l), i \neq j \neq k \neq l$
-6	$\frac{n(n-1)(n+1)(n+6)}{24}$	$\sum_{\sigma \in \mathfrak{S}_4} e[i, j, k, l]^\sigma, i < j < k < l$ $e[i, i, i, i] + e[j, j, j, j] - \sum'_{\sigma \in \mathfrak{S}_4} e[i, i, j, j]^\sigma, i < j$

The star  $\star$  means that the eigenvectors of a basis are listed up to action of  $\mathfrak{S}_4$ ; and the symbols  $\sum'_{\sigma \in \mathfrak{S}_4}$  mean that we take the sum of all *distinct* permutations of the tensors. For the eigenvectors associated to the value 2, denote  $\mathfrak{D}_{4,(1)} = \langle (1, 3, 2, 4), (1, 2) \rangle$ ,  $\mathfrak{D}_{4,(2)} = \langle (1, 2, 3, 4), (1, 3) \rangle$  and  $\mathfrak{D}_{4,(3)} = \langle (1, 2, 4, 3), (1, 4) \rangle$  the three dihedral groups of order 4 (hence cardinality 8) that can be found inside  $\mathfrak{S}_4$ . Each dihedral group of order 4 has for presentation

$$\mathfrak{D}_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle,$$

so the parity  $\eta(\sigma)$  of the number of occurrences of  $s$  in a reduced writing of  $\sigma \in \mathfrak{D}_4$  is well-defined, and provides a morphism  $\eta : \mathfrak{D}_{4,(v)} \rightarrow \{\pm 1\}$  for  $v = 1, 2, 3$ . Then, it can be checked that for every  $i \neq j \neq k \neq l$  and any  $v$ ,

$$D_v^\eta(i, j, k, l) = \sum_{\sigma \in \mathfrak{D}_{4,(v)}} \eta(\sigma) e[i, j, k, l]^\sigma$$

is in  $V_2$ . The eigenvectors  $D_1^\theta(i, j, k, l)$ ,  $D_2^\theta(i, j, k, l)$  and  $D_3^\theta(i, j, k, l)$  associated to the eigenvalue  $-2$  are defined exactly the same way, but with the morphism  $\theta : \mathfrak{D}_{4,(v)} \rightarrow \{\pm 1\}$  associated to the parity of the number of occurrences of  $r$  in a reduced decomposition of  $\sigma \in \mathfrak{D}_4$  (again it is well defined):

$$D_v^\theta(i, j, k, l) = \sum_{\theta \in \mathfrak{D}_{4,(v)}} \theta(\sigma) e[i, j, k, l]^\sigma.$$

For the eigenvectors associated to the value 0,  $S_1(i, j, k, l)$  is defined by

$$\begin{aligned} & e[i, i, k, k] + e[k, k, i, i] + e[j, j, l, l] + e[l, l, j, j] - e[i, i, l, l] - e[l, l, i, i] - e[j, j, k, k] - e[k, k, j, j] \\ & - e[i, k, k, i] - e[k, i, i, k] - e[j, l, l, j] - e[l, j, j, l] + e[i, l, l, i] + e[l, i, i, l] + e[j, k, k, j] + e[k, j, j, k] \end{aligned}$$

and  $S_2(i, j, k, l)$  is obtained by replacing each term  $a \otimes b \otimes b \otimes a$  by  $a \otimes b \otimes a \otimes b$  in the previous formula. On the other hand, if  $\mathfrak{K}_4 = \{\text{id}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$  denotes the Klein group, then  $K_1(i, j, k, l)$



and  $K_2(i, j, k, l)$  are defined as

$$\begin{aligned} K_1(i, j, k, l) &= \sum_{\sigma \in \mathfrak{R}_4} e[i, j, k, l]^\sigma - \sum_{\sigma \in (1,2,3)\mathfrak{R}_4} e[i, j, k, l]^\sigma; \\ K_2(i, j, k, l) &= \sum_{\sigma \in \mathfrak{R}_4} e[i, j, k, l]^\sigma - \sum_{\sigma \in (1,3,2)\mathfrak{R}_4} e[i, j, k, l]^\sigma. \end{aligned}$$

That said, the deduction of the mixed moments of order 4 of the coefficients of  $g$  goes as follows. One notices that

$$(n+2) \sum_{i=1}^n e_i^{\otimes 4} = \left( \sum_{k,l=1}^n e[k, k, l, l] + e[k, l, k, l] + e[k, l, l, k] \right) + \sum_{i < j} \left( e_i^{\otimes 4} + e_j^{\otimes 4} - \sum_{\sigma \in \mathfrak{S}_4} e[i, i, j, j]^\sigma \right)$$

with the first sum in the eigenspace  $V_{2n-2}$  and the second sum in the eigenspace  $V_{-6}$ . On the other hand, for any  $i \neq j$ ,

$$\begin{aligned} (n+4)(e_i^{\otimes 4} - e_j^{\otimes 4}) &= 6e[i, i, i, i] + \sum_{k \neq i, j} \sum_{\sigma \in \mathfrak{S}_4} e[i, i, k, k]^\sigma - \sum_{k \neq i, j} \sum_{\sigma \in \mathfrak{S}_4} e[j, j, k, k]^\sigma - 6e[j, j, j, j] \\ &+ \sum_{k \neq i, j} \left( e[i, i, i, i] + e[k, k, k, k] - \sum_{\sigma \in \mathfrak{S}_4} e[i, i, k, k]^\sigma \right) - \left( e[j, j, j, j] + e[k, k, k, k] - \sum_{\sigma \in \mathfrak{S}_4} e[j, j, k, k]^\sigma \right), \end{aligned}$$

with the first line in  $V_{n-2}$  and the second in  $V_{-6}$ . Since  $e_i^{\otimes 4} = \frac{1}{n} \sum_{j=1}^n e_j^{\otimes 4} + \frac{1}{n} \sum_{j=1}^n (e_i^{\otimes 4} - e_j^{\otimes 4})$ , one concludes that

$$\begin{aligned} e_i^{\otimes 4} &= \frac{1}{n(n+2)} \sum_{k,l=1}^n e[k, k, l, l] + e[k, l, k, l] + e[k, l, l, k] \\ &+ \frac{1}{n(n+4)} \sum_{\sigma \in \mathfrak{S}_4} \sum_{k,l=1}^n (e[i, i, k, k] - e[l, l, k, k])^\sigma \\ &+ \frac{n+1}{(n+2)(n+4)} \sum_{k \neq i} \left( e[i, i, i, i] + e[k, k, k, k] - \sum_{\sigma \in \mathfrak{S}_4} e[i, i, k, k]^\sigma \right) \\ &- \frac{1}{(n+2)(n+4)} \sum_{(k < l) \neq i} \left( e[k, k, k, k] + e[l, l, l, l] - \sum_{\sigma \in \mathfrak{S}_4} e[k, k, l, l]^\sigma \right), \end{aligned}$$

each line being in a different eigenspace:  $V_{2n-2}$ ,  $V_{n-2}$ ,  $V_{-6}$  and  $V_{-6}$ . This leads to the following formulas<sup>16</sup>:

$$\begin{aligned} \mathbb{E}[(g_{ii})^4] &= \frac{3}{n(n+2)} + \frac{6(n-1)}{n(n+4)} e^{-t} + \frac{(n+1)(n-1)}{(n+2)(n+4)} e^{-\frac{2n+4}{n}t}, \\ \mathbb{E}[(g_{ij})^4] &= \frac{3}{n(n+2)} - \frac{6}{n(n+4)} e^{-t} + \frac{3}{(n+2)(n+4)} e^{-\frac{2n+4}{n}t}, \\ \mathbb{E}[(g_{ii})^2(g_{ij})^2] &= \frac{1}{n(n+2)} + \frac{(n-2)}{n(n+4)} e^{-t} - \frac{(n+1)}{(n+2)(n+4)} e^{-\frac{2n+4}{n}t}, \\ \mathbb{E}[(g_{ij})^2(g_{ik})^2] &= \frac{1}{n(n+2)} - \frac{2}{n(n+4)} e^{-t} + \frac{1}{(n+2)(n+4)} e^{-\frac{2n+4}{n}t}, \end{aligned}$$

<sup>16</sup>The technique is the following: to compute  $\mathbb{E}[g_{ij_1} g_{ij_2} g_{ij_3} g_{ij_4}]$ , one counts the number of occurrences of  $e[j_1, j_2, j_3, j_4]$  in each term of the previous expansion.

and one sees also that the other expectations  $\mathbb{E}[g_{ij}g_{ik}g_{il}g_{im}]$  vanish (e.g.,  $\mathbb{E}[g_{ij}g_{ik}(g_{il})^2]$  with  $i \neq j \neq k \neq l$ ). Similar manipulations yield the decomposition in eigenvectors of  $e_i^{\otimes 2} \otimes e_j^{\otimes 2}$ :

$$\begin{aligned}
& \frac{n+1}{n(n-1)(n+2)} \sum_{k,l=1}^n e[k,k,l,l] - \frac{1}{(n-1)n(n+2)} \sum_{k,l=1}^n e[k,l,k,l] + e[k,l,l,k] \\
& \dots \\
& + \frac{1}{n(n-2)} \sum_{k \neq i,j} \left( \sum_{l=1}^n e[i,i,l,l] - e[k,k,l,l] + \sum_{l=1}^n e[l,l,j,j] - e[l,l,k,k] \right) \\
& - \frac{1}{n(n-2)(n+4)} \sum_{k \neq i,j} \sum_{\sigma \in \mathfrak{S}_4} \left( \sum_{l=1}^n (e[i,i,l,l] - e[k,k,l,l])^\sigma + \sum_{l=1}^n (e[l,l,j,j] - e[l,l,k,k])^\sigma \right) \\
& \dots \\
& + \frac{1}{6(n-1)(n-2)} \sum_{(k < l) \neq i,j} S_1(i,k,j,l) + S_1(i,l,j,k) + S_2(i,k,j,l) + S_2(i,l,j,k) \\
& \dots \\
& + \frac{1}{2n} \sum_{k \neq i,j} e[i,i,j,j] + e[j,j,k,k] + e[k,k,i,i] - e[j,j,i,i] - e[k,k,j,j] - e[i,i,k,k] \\
& \dots \\
& + \frac{1}{6(n+4)} \left( \sum_{k \neq i} \left( e_i^{\otimes 4} + e_k^{\otimes 4} - \sum_{\sigma \in \mathfrak{S}_4} e[i,i,k,k]^\sigma \right) + \sum_{k \neq j} \left( e_j^{\otimes 4} + e_k^{\otimes 4} - \sum_{\sigma \in \mathfrak{S}_4} e[j,j,k,k]^\sigma \right) \right) \\
& - \frac{1}{6} \left( e_i^{\otimes 4} + e_j^{\otimes 4} - \sum_{\sigma \in \mathfrak{S}_4} e[i,i,j,j]^\sigma \right) - \frac{1}{3(n+2)(n+4)} \sum_{(k < l)} \left( e_k^{\otimes 4} + e_l^{\otimes 4} - \sum_{\sigma \in \mathfrak{S}_4} e[k,k,l,l]^\sigma \right),
\end{aligned}$$

where the eigenspaces associated to each part are  $V_{2n-2}$ ,  $V_{n-2}$ ,  $V_0$ ,  $V_{-2}$  and  $V_{-6}$ . As a consequence,

$$\begin{aligned}
\mathbb{E}[(g_{ii})^2(g_{jj})^2] &= \frac{n+1}{(n-1)n(n+2)} + \frac{2(n+3)}{n(n+4)} e^{-t} + \frac{n-3}{3(n-1)} e^{-\frac{2n-2}{n}t} + \frac{n-2}{2n} e^{-2t} \\
&+ \frac{n^2+4n+6}{6(n+2)(n+4)} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[(g_{ij})^2(g_{ji})^2] &= \frac{n+1}{n(n-1)(n+2)} - \frac{2}{n(n+4)} e^{-t} + \frac{n-3}{3(n-1)} e^{-\frac{2n-2}{n}t} - \frac{n-2}{2n} e^{-2t} \\
&+ \frac{n^2+4n+6}{6(n+2)(n+4)} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[(g_{ii})^2(g_{jk})^2] &= \frac{n+1}{n(n-1)(n+2)} + \frac{n^2-8}{n(n-2)(n+4)} e^{-t} - \frac{n-3}{3(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&- \frac{1}{2n} e^{-2t} - \frac{n}{6(n+2)(n+4)} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[(g_{ij})^2(g_{jk})^2] &= \frac{n+1}{n(n-1)(n+2)} - \frac{2}{(n-2)(n+4)} e^{-t} - \frac{n-3}{3(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&+ \frac{1}{2n} e^{-2t} - \frac{n}{6(n+2)(n+4)} e^{-\frac{2n+4}{n}t};
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[(g_{ij})^2(g_{kl})^2] &= \frac{n+1}{n(n-1)(n+2)} - \frac{2(n+2)}{n(n-2)(n+4)} e^{-t} + \frac{2}{3(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{1}{3(n+2)(n+4)} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[g_{ii}g_{ij}g_{jj}g_{ji}] &= -\frac{1}{n(n-1)(n+2)} - \frac{2}{n(n+4)} e^{-t} - \frac{n-3}{6(n-1)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{n^2+4n+6}{6(n+2)(n+4)} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[g_{ik}g_{il}g_{jk}g_{jl}] &= -\frac{1}{n(n-1)(n+2)} + \frac{4}{n(n-2)(n+4)} e^{-t} - \frac{1}{3(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{1}{3(n+2)(n+4)} e^{-\frac{2n+4}{n}t}.
\end{aligned}$$

Finally, the elementary tensor  $e_i \otimes e_j \otimes e_k \otimes e_l$  with  $i \neq j \neq k \neq l$  can be expanded as a combination of eigenvectors in  $V_6, V_2, V_0, V_{-2}$  and  $V_{-6}$ . This expansion is related to the following remarkable identity in the group algebra  $\mathbb{C}\mathfrak{S}_4$ , which can be considered as a relation of orthogonality of characters, but that only involves one-dimensional representations. Denote

$$D_1^\eta = \sum_{\sigma \in \mathfrak{D}_{4,(1)}} \eta(\sigma) \sigma,$$

and similarly for  $D_2^\eta, D_3^\eta, D_1^\theta, D_2^\theta$  and  $D_3^\theta$ . We also introduce  $I = \sum_{\sigma \in \mathfrak{S}_4} \sigma$ ,  $E = \sum_{\sigma \in \mathfrak{S}_4} \varepsilon(\sigma) \sigma$ , and

$$K_1 = \sum_{\sigma \in \mathfrak{K}_4} \sigma - \sum_{\sigma \in (1,2,3)\mathfrak{K}_4} \sigma \quad ; \quad K_2 = \sum_{\sigma \in \mathfrak{K}_4} \sigma - \sum_{\sigma \in (1,3,2)\mathfrak{K}_4} \sigma.$$

As explained before, all these sums correspond to eigenvectors in  $V_6, V_2, V_0, V_{-2}$  and  $V_{-6}$ . Then,

$$\text{id}_{[1,4]} = \frac{1}{24} I + \frac{1}{8} (D_1^\eta + D_2^\eta + D_3^\eta) + \frac{1}{12} (K_1 + K_2) + \frac{1}{8} (D_1^\theta + D_2^\theta + D_3^\theta) + \frac{1}{24} E. \quad (4.9)$$

Indeed,  $\frac{1}{24} I + \frac{1}{24} E$  is one twelfth of the sum of all even permutations in the alternate group  $\mathfrak{A}_4$ . By adding  $\frac{1}{12} (K_1 + K_2)$  to this quantity, one removes all the permutations that are in  $\mathfrak{A}_4$  and not in  $\mathfrak{K}_4$ , so

$$\frac{1}{24} I + \frac{1}{12} (K_1 + K_2) + \frac{1}{24} E = \frac{1}{4} \sum_{\sigma \in \mathfrak{K}_4} \sigma.$$

On the other hand,

$$\frac{1}{8} (D_1^\eta + D_1^\theta) = \frac{1}{4} \text{id}_{[1,4]} + \frac{1}{4} (1,2)(3,4) - \frac{1}{4} (1,3)(2,4) - \frac{1}{4} (1,4)(2,3)$$

and similarly for the two other dihedral groups. Thus, the whole sum on the right-hand side of (4.9) is indeed  $\text{id}_{[1,4]}$ . As a consequence,  $e[i, j, k, l]$  is equal to

$$\begin{aligned}
&\frac{1}{24} \sum_{\sigma \in \mathfrak{S}_4} \varepsilon(\sigma) e[i, j, k, l]^\sigma + \frac{1}{8} (D_1^\eta(i, j, k, l) + D_2^\eta(i, j, k, l) + D_3^\eta(i, j, k, l)) + \frac{1}{12} (K_1(i, j, k, l) + K_2(i, j, k, l)) \\
&+ \frac{1}{8} (D_1^\theta(i, j, k, l) + D_2^\theta(i, j, k, l) + D_3^\theta(i, j, k, l)) + \frac{1}{24} \sum_{\sigma \in \mathfrak{S}_4} e[i, j, k, l]^\sigma
\end{aligned}$$

with each term respectively in  $V_6, V_2, V_0, V_{-2}$  and  $V_{-6}$ . This leads to

$$\begin{aligned}\mathbb{E}[g_{ii}g_{jj}g_{kk}g_{ll}] &= \frac{1}{24}e^{-\frac{2n-8}{n}t} + \frac{3}{8}e^{-\frac{2n-4}{n}t} + \frac{1}{6}e^{-\frac{2n-2}{n}t} + \frac{3}{8}e^{-2t} + \frac{1}{24}e^{-\frac{2n+4}{n}t}, \\ \mathbb{E}[g_{ij}g_{jk}g_{kl}g_{li}] &= -\frac{1}{24}e^{-\frac{2n-8}{n}t} + \frac{1}{8}e^{-\frac{2n-4}{n}t} - \frac{1}{8}e^{-2t} + \frac{1}{24}e^{-\frac{2n+4}{n}t}, \\ \mathbb{E}[g_{ii}g_{jj}g_{kl}g_{lk}] &= -\frac{1}{24}e^{-\frac{2n-8}{n}t} - \frac{1}{8}e^{-\frac{2n-4}{n}t} + \frac{1}{8}e^{-2t} + \frac{1}{24}e^{-\frac{2n+4}{n}t}, \\ \mathbb{E}[g_{ij}g_{ji}g_{kl}g_{lk}] &= \frac{1}{24}e^{-\frac{2n-8}{n}t} - \frac{1}{8}e^{-\frac{2n-4}{n}t} + \frac{1}{6}e^{-\frac{2n-2}{n}t} - \frac{1}{8}e^{-2t} + \frac{1}{24}e^{-\frac{2n+4}{n}t},\end{aligned}$$

and we are done with the computations in the case of special orthogonal groups.

*Remark.* It would be very nice to find the analogue of Equation (4.9) for symmetric groups of larger order, in connection with the diagonalization of  $M_{n,2k}$ . An interesting feature of this identity is that it involves only generating functions of one-dimensional characters of subgroups  $H \subset \mathfrak{S}_4$ :

$$\Sigma(H, \chi) = \sum_{\sigma \in H \subset \mathfrak{S}_4} \chi(\sigma) \sigma.$$

Indeed, this is obvious for most of the terms, and notice that  $K_1 + K_2 = \Sigma(\mathfrak{A}_4, \chi) + \Sigma(\mathfrak{A}_4, \chi^2)$ , where  $\chi$  is the quotient map  $\mathfrak{A}_4 \rightarrow \mathfrak{A}_4/\mathfrak{K}_4 \simeq \mathbb{Z}/3\mathbb{Z} = \{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$ . This combinatorial property of the reduction of  $M_{n,2k}$  seems profound and quite mysterious for the moment.

**Proposition 28.** *For the real Grassmannian varieties  $\text{Gr}(n, q, \mathbb{R})$  and the spaces  $\text{SO}(2n)/\text{U}(n)$ , the coefficients of Lemma 26 are the following:*

$$\begin{aligned}\text{Gr}(n, q, \mathbb{R}) : & \quad \frac{2}{n^2 + n - 2} + \frac{2n^2}{3} \left( \frac{1}{(n-1)(n-2)} - \frac{1}{pq(n-2)} \right) \phi^{(2,2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} \\ & + \frac{\frac{4n^2}{pq} - 16}{(n-2)(n+4)} \phi^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} + \frac{n^2}{3} \left( \frac{1}{(n+2)(n+4)} + \frac{2}{pq(n+4)} \right) \phi^{(4,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}}; \\ \text{SO}(2n)/\text{U}(n) : & \quad \frac{1}{2n^2 - n} + \frac{n-1}{3n} \phi^{(1^4,0,\dots,0)_n} + \frac{4(n^2-1)}{(3n)(2n-1)} \phi^{(2,2,0,\dots,0)_n}.\end{aligned}$$

*Proof.* One expands the square of the sum given by Proposition 25, and one gathers the joint moments of the coefficients according to the possible identities between the indices. For real Grassmannians,  $(\phi^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}})^2$  has for expansion:

$$\begin{aligned}& \left( \frac{1}{p} \sum_{i,j \leq p} (g_{ij})^2 + \frac{1}{q} \sum_{i,j > p} (g_{ij})^2 - 1 \right)^2 = \left( \frac{1}{p} \sum_{i,j \leq p} (g_{ij})^2 + \frac{1}{q} \sum_{i,j > p} (g_{ij})^2 \right)^2 - 2 \phi^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} - 1 \\ &= \left( \frac{n}{pq} \right) T[(g_{11})^4] + \left( 4 - \frac{n}{pq} \right) T[(g_{11}g_{22})^2] + \left( 2 - \frac{n}{pq} \right) (4T[(g_{11}g_{12})^2] + T[(g_{12})^4] + T[(g_{12}g_{21})^2]) \\ &+ \left( 4n - 16 + \frac{4n}{pq} \right) T[(g_{11}g_{23})^2] + \left( 2n - 12 + \frac{4n}{pq} \right) (T[(g_{12}g_{13})^2] + T[(g_{12}g_{23})^2]) \\ &+ \left( n^2 - 8n + 24 - \frac{6n}{pq} \right) T[(g_{12}g_{34})^2] - 2 \phi^{(2,0,\dots,0)_{\lfloor \frac{n}{2} \rfloor}} - 1.\end{aligned}$$

where by  $T[(g_{11})^4]$  we mean a linear combination of products  $(g_{ii})^4$ , whose expectation is therefore  $\mathbb{E}[(g_{11})^4]$ ; by  $T[(g_{11}g_{22})^2]$  we mean a linear combination of products  $(g_{ii}g_{jj})^2$  whose expectation will be  $\mathbb{E}[(g_{11}g_{22})^2]$ , etc.

Thus, the expectation of  $\left(\phi^{(2,0,\dots,0)}_{\lfloor \frac{n}{2} \rfloor}\right)^2$  is

$$\begin{aligned} & \frac{2}{n^2 + n - 2} + \frac{\frac{4n^2}{pq} - 16}{(n-2)(n+4)} e^{-t} + \frac{2n^2}{3} \left( \frac{1}{(n-1)(n-2)} - \frac{1}{pq(n-2)} \right) e^{-\frac{2n-2}{n}t} \\ & + \frac{n^2}{3} \left( \frac{1}{(n+2)(n+4)} + \frac{2}{pq(n+4)} \right) e^{-\frac{2n+4}{n}t}, \end{aligned}$$

and by identifying the Casimir coefficients of the spherical functions, one deduces from this the expansion of the square of the discriminating zonal function in zonal functions.

For the spaces  $\mathrm{SO}(2n)/\mathrm{U}(n)$ , one computes again the square of the homogeneous polynomial of degree 2 given by Proposition 25:

$$\begin{aligned} & \frac{1}{n^2} \left( \sum_{i,j=1}^n g_{2i,2j} g_{2i-1,2j-1} - g_{2i,2j-1} g_{2i-1,2j} \right)^2 \\ & = \frac{1}{n} (T[(g_{11}g_{22})^2] + T[(g_{12}g_{21})^2] - 2T[g_{11}g_{12}g_{22}g_{21}]) \\ & \quad + \frac{n-1}{n} (2T[(g_{12}g_{34})^2] - 2T[g_{13}g_{14}g_{23}g_{24}]) \\ & \quad + \frac{n-1}{n} (2T[g_{12}g_{21}g_{34}g_{43}] - 2T[g_{12}g_{23}g_{34}g_{41}]) \\ & \quad + \frac{n-1}{n} (T[g_{11}g_{22}g_{33}g_{44}] + T[g_{12}g_{21}g_{34}g_{43}] - 2T[g_{11}g_{22}g_{34}g_{43}]) \\ & \quad + \text{remainder}, \end{aligned}$$

with the same notations as before, and where the remainder is a combination of products of coefficients whose expectation vanish under Brownian (and Haar) measures. More precisely, terms with a certain symmetry cancel with one another when taking the expectation: for instance,

$$(g_{2i,2j} g_{2i-1,2j-1} - g_{2i,2j-1} g_{2i-1,2j}) \times (g_{2k,2l} g_{2k-1,2l-1} - g_{2k,2l-1} g_{2k-1,2l}) \quad (4.10)$$

with  $i \neq j \neq k \neq l$  is equal to  $a + b - c - d$ , where  $a, b, c, d$  are products of type  $g_{ij}g_{kl}g_{mn}g_{op}$ , and have therefore the same expectation. Consequently, every product of type (4.10) will not contribute to the expectation of  $(\phi^{(1,1,0,\dots,0)}_{\lfloor \frac{n}{2} \rfloor})^2$ . The following sets of indices have the same property of “self-cancellation”:

$$\begin{aligned} & (i, i, i, j) ; (i, i, j, i) ; (i, j, i, i) ; (j, i, i, i) ; (i, i, j, k) ; (j, k, i, i) ; \\ & (i, j, i, k) ; (j, i, k, i) ; (i, j, k, i) ; (j, i, i, k) ; (i, j, k, l) ; (i, j, k, l) ; \end{aligned}$$

so it suffices to consider products with sets of indices  $(i, i, i, i)$ ,  $(i, j, i, j)$ ,  $(i, j, j, i)$  or  $(i, i, j, j)$  — these are the four lines of the previous expansion. Using the formulas given before for the joint moments of the entries (beware that one has to use them with the parameter  $2n$ ), we obtain:

$$\mathbb{E}[(\phi^{(1,1,0,\dots,0)}_n)^2] = \frac{1}{n(2n-1)} + \frac{n-1}{3n} e^{-\frac{2n-4}{n}t} + \frac{4(n^2-1)}{(3n)(2n-1)} e^{-\frac{2n-1}{n}t}$$

and it suffices then to identify the coefficients of the negative exponentials.  $\square$

**4.1.3. Quotients of unitary groups.** For special unitary groups, set  $\frac{1}{n^2}M_{n,k,l} = \sum_{1 \leq i < j \leq k+l} \tilde{\eta}_{i,j}(C_{\mathrm{su}(n)})$ , viewed as an element of  $\mathrm{End}((\mathbb{C}^n)^{\otimes k+l})$ . If  $k = l = 1$ , then the eigenvalues and eigenvectors of  $M_{n,1,1} = iI_n \otimes iI_n + n \sum_{i,j=1}^n E_{ij} \otimes E_{ij}$  are:

eigenvalue	multiplicity	eigenvectors
$n^2 - 1$	1	$\sum_{i=1}^n e[i, i]$
-1	$n^2 - 1$	$e[i, j], 1 \leq i \neq j \leq n$ $e[i, i] - e[i + 1, i + 1], i \leq n - 1$

Consequently,

$$\mathbb{E}[|g_{ii}|^2] = \frac{1}{n} + \left(1 - \frac{1}{n}\right) e^{-t} \quad ; \quad \mathbb{E}[|g_{ij}|^2] = \frac{1}{n} (1 - e^{-t}) \quad ; \quad \mathbb{E}[g_{ii} \overline{g_{jj}}] = e^{-t}.$$

If  $k = l = 2$ , then the eigenvalues and eigenvectors of  $M_{n,2,2}$  are:

eigenvalue	multiplicity	eigenvectors (not exhaustive, some repetitions)
$2n^2 - 2$	2	$\sum_{k=1}^n \sum_{l=1}^n e[k, l, k, l], \sum_{k=1}^n \sum_{l=1}^n e[k, l, l, k]$
$n^2 - 2$	$4(n+1)(n-1)$	$\sum_{k=1}^n e[i, k, i, k] - e[i+1, k, i+1, k], i \leq n-1$ $\sum_{k=1}^n e[k, i, k, i] - e[k, i+1, k, i+1], i \leq n-1$ $\sum_{k=1}^n e[i, k, k, i] - e[i+1, k, k, i+1], i \leq n-1$ $\sum_{k=1}^n e[k, i, i, k] - e[k, i+1, i+1, k], i \leq n-1$
$2n - 2$	$\frac{n^2(n+1)(n-3)}{4}$	$\left( (e[i, j] - e[j, i])^{\otimes 2} - (e[j, k] - e[k, j])^{\otimes 2} \right), i \neq j \neq k \neq l$ $\left( + (e[k, l] - e[l, k])^{\otimes 2} - (e[l, i] - e[i, l])^{\otimes 2} \right)$
-2	$\frac{(n+2)(n+1)(n-1)(n-2)}{2}$	$\left( \begin{matrix} e[i, j, i, j] + e[j, k, j, k] + e[k, i, k, i] \\ -e[j, i, j, i] - e[k, j, k, j] - e[i, k, i, k] \end{matrix} \right), i < j < k$ $\left( \begin{matrix} e[i, j, j, i] + e[j, k, k, j] + e[k, i, i, k] \\ -e[j, i, i, j] - e[k, j, j, k] - e[i, k, k, i] \end{matrix} \right), i < j < k$
$-2n - 2$	$\frac{n^2(n-1)(n+3)}{4}$	$e[i, i, i, i] + e[j, j, j, j] - (e[i, j] + e[j, i])^{\otimes 2}, i < j$

For the eigenvectors associated to the value  $2n - 2$ , we shall write

$$S(i, j, k, l) = (e[i, j] - e[j, i])^{\otimes 2} - (e[j, k] - e[k, j])^{\otimes 2} + (e[k, l] - e[l, k])^{\otimes 2} - (e[l, i] - e[i, l])^{\otimes 2}.$$

Again, we can use the previous table to decompose some elementary 4-tensors in eigenvectors of  $M_{n,2,2}$ . Thus,  $e_i^{\otimes 4}$  is equal to

$$\begin{aligned} & \frac{1}{n(n+1)} \sum_{k,l=1}^n e[k, l, k, l] + e[k, l, l, k] + \frac{1}{n(n+2)} \sum_{k \neq i} \sum_{l=1}^n \left( \begin{matrix} (e[i, l, i, l] - e[k, l, k, l]) + (e[l, i, l, i] - e[l, k, l, k]) \\ + (e[i, l, l, i] - e[k, l, l, k]) + (e[l, i, i, l] - e[l, k, k, l]) \end{matrix} \right) \\ & + \frac{1}{n+2} \sum_{k \neq i} e_i^{\otimes 4} + e_k^{\otimes 4} - (e[i, k] + e[k, i])^{\otimes 2} - \frac{1}{(n+1)(n+2)} \sum_{k < l} e_k^{\otimes 4} + e_l^{\otimes 4} - (e[k, l] + e[l, k])^{\otimes 2} \end{aligned}$$

with the two first terms respectively in  $V_{2n^2-2}$  and  $V_{n^2-2}$ , and the second line in  $V_{-2n-2}$ . It follows that

$$\begin{aligned}\mathbb{E}[|g_{ii}|^4] &= \frac{2}{n(n+1)} + \frac{4(n-1)}{n(n+2)} e^{-t} + \frac{n(n-1)}{(n+1)(n+2)} e^{-\frac{2n+2}{n}t}; \\ \mathbb{E}[|g_{ij}|^4] &= \frac{2}{n(n+1)} - \frac{4}{n(n+2)} e^{-t} + \frac{2}{(n+1)(n+2)} e^{-\frac{2n+2}{n}t}; \\ \mathbb{E}[|g_{ii}|^2|g_{ij}|^2] &= \frac{1}{n(n+1)} + \frac{n-2}{n(n+2)} e^{-t} - \frac{n}{(n+1)(n+2)} e^{-\frac{2n+2}{n}t}; \\ \mathbb{E}[|g_{ij}|^2|g_{ik}|^2] &= \frac{1}{n(n+1)} - \frac{2}{n(n+2)} e^{-t} + \frac{1}{(n+1)(n+2)} e^{-\frac{2n+2}{n}t}.\end{aligned}$$

Similarly,  $e_i \otimes e_j \otimes e_i \otimes e_j$  is equal to

$$\begin{aligned}& \frac{1}{(n-1)(n+1)} \sum_{k=1}^n \sum_{l=1}^n e[k, l, k, l] - \frac{1}{n(n-1)(n+1)} \sum_{k=1}^n \sum_{l=1}^n e[k, l, l, k] \\ & \dots \\ & + \frac{1}{n(n+2)} \left( \sum_{l=1}^n \left( \frac{e[i, l, i, l] - e[j, l, j, l]}{+e[l, j, l, j] - e[l, i, i, i]} \right) \right) + \frac{1}{(n-2)(n+2)} \sum_{k \neq i, j} \left( \sum_{l=1}^n \left( \frac{e[i, l, i, l] - e[k, l, k, l]}{+e[l, j, j, j] - e[l, k, k, k]} \right) \right) \\ & - \frac{1}{n(n-2)(n+2)} \sum_{k \neq i, j} \left( \sum_{l=1}^n \left( \frac{e[i, l, l, i] + e[j, l, l, j] - 2e[k, l, l, k]}{+e[l, i, i, i] + e[l, j, j, j] - 2e[l, k, k, l]} \right) \right) \\ & \dots \\ & + \frac{1}{4(n-1)(n-2)} \sum_{(k < l) \neq i, j} 2S(i, j, k, l) - S(i, k, j, l) \\ & \dots \\ & + \frac{1}{2n} \sum_{k \neq i, j} \left( \frac{e[i, j, i, j] + e[j, k, j, k] + e[k, i, k, i]}{-e[j, i, j, i] - e[k, j, k, j] - e[i, k, i, k]} \right) \\ & \dots \\ & + \frac{1}{4(n+2)} \left( \sum_{k \neq i} e_i^{\otimes 4} + e_k^{\otimes 4} - (e[i, k] + e[k, i])^{\otimes 2} + \sum_{k \neq j} e_j^{\otimes 4} + e_k^{\otimes 4} - (e[j, k] + e[k, j])^{\otimes 2} \right) \\ & - \frac{1}{4} (e_i^{\otimes 4} + e_j^{\otimes 4} - (e[i, j] + e[j, i])^{\otimes 2}) - \frac{1}{2(n+1)(n+2)} \left( \sum_{k < l} e_k^{\otimes 4} + e_l^{\otimes 4} - (e[k, l] + e[l, k])^{\otimes 2} \right)\end{aligned}$$

with the parts of this expansion respectively in  $V_{2n^2-2}$ ,  $V_{n^2-2}$ ,  $V_{2n-2}$ ,  $V_{-2}$  and  $V_{-2n-2}$ . This yields

$$\begin{aligned}\mathbb{E}[|g_{ii}|^2|g_{jj}|^2] &= \frac{1}{(n-1)(n+1)} + \frac{2(n+1)}{n(n+2)} e^{-t} + \frac{n-3}{4(n-1)} e^{-\frac{2n-2}{n}t} \\ & \quad + \frac{n-2}{2n} e^{-2t} + \frac{n^2+n+2}{4(n+1)(n+2)} e^{-\frac{2n+2}{n}t}; \\ \mathbb{E}[|g_{ij}|^2|g_{ji}|^2] &= \frac{1}{(n-1)(n+1)} - \frac{2}{n(n+2)} e^{-t} + \frac{n-3}{4(n-1)} e^{-\frac{2n-2}{n}t} \\ & \quad - \frac{n-2}{2n} e^{-2t} + \frac{n^2+n+2}{4(n+1)(n+2)} e^{-\frac{2n+2}{n}t};\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[|g_{ii}|^2 |g_{jk}|^2] &= \frac{1}{(n-1)(n+1)} + \frac{n^2 - 2n - 2}{n(n-2)(n+2)} e^{-t} - \frac{n-3}{4(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad - \frac{1}{2n} e^{-2t} - \frac{n-1}{4(n+1)(n+2)} e^{-\frac{2n+2}{n}t}, \\
\mathbb{E}[|g_{ij}|^2 |g_{jk}|^2] &= \frac{1}{(n-1)(n+1)} - \frac{2(n-1)}{n(n-2)(n+2)} e^{-t} - \frac{n-3}{4(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{1}{2n} e^{-2t} - \frac{n-1}{4(n+1)(n+2)} e^{-\frac{2n+2}{n}t}, \\
\mathbb{E}[|g_{ij}|^2 |g_{kl}|^2] &= \frac{1}{(n-1)(n+1)} - \frac{2}{(n-2)(n+2)} e^{-t} + \frac{1}{2(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{1}{2(n+1)(n+2)} e^{-\frac{2n+2}{n}t}.
\end{aligned}$$

**Proposition 29.** *For the symmetric spaces with isometry group  $\mathrm{SU}(n)$  or  $\mathrm{SU}(2n)$ , the coefficients of Lemma 26 are the following:*

$$\begin{aligned}
\mathrm{Gr}(n, q, \mathbb{C}) : \quad & \frac{1}{n^2 - 1} + \frac{\frac{2n^2}{pq} - 8}{n^2 - 4} \phi^{(2,1,\dots,1)_{n-1}} + \frac{n^2}{2} \left( \frac{1}{(n-1)(n-2)} - \frac{1}{pq(n-2)} \right) \phi^{(2,2,1,\dots,1,0)_{n-1}}, \\
& + \frac{n^2}{2} \left( \frac{1}{(n+1)(n+2)} + \frac{1}{pq(n+2)} \right) \phi^{(4,2,\dots,2)_{n-1}}, \\
\mathrm{SU}(n)/\mathrm{SO}(n) : \quad & \frac{2}{n^2 + n} + \frac{n^2 + n - 2}{n^2 + n} \phi^{(4,2,\dots,2)_{n-1}}, \\
\mathrm{SU}(2n)/\mathrm{USp}(n) : \quad & \frac{1}{2n^2 - n} + \frac{2n^2 - n - 1}{2n^2 - n} \phi^{(2,2,1,\dots,1,0)_{2n-1}}.
\end{aligned}$$

*Proof.* For  $\mathrm{SU}(n)/\mathrm{SO}(n)$  and  $\mathrm{SU}(2n)/\mathrm{USp}(n)$ , the only missing coefficient has already been computed. For complex Grassmannians,  $(\phi^{(2,1,\dots,1)_{n-1}})^2$  has exactly the same expansion as in the real case, but with square modules. From the computation of the joint moments  $\mathbb{E}[|g_{ij}g_{kl}|^2]$  performed previously, one deduces that the expectation of the square of the discriminating zonal function is

$$\begin{aligned}
& \frac{1}{n^2 - 1} + \frac{\frac{2n^2}{pq} - 8}{n^2 - 4} e^{-t} + \frac{n^2}{2} \left( \frac{1}{(n-1)(n-2)} - \frac{1}{pq(n-2)} \right) e^{-\frac{2n-2}{n}t} \\
& + \frac{n^2}{2} \left( \frac{1}{(n+1)(n+2)} + \frac{1}{pq(n+2)} \right) e^{-\frac{2n+2}{n}t}
\end{aligned}$$

whence the expansion in zonal spherical functions by identifying the coefficients.  $\square$

**4.1.4. Quotients of symplectic groups.** Finally, set  $\frac{1}{n}M_{n,k} = \sum_{1 \leq i < j \leq k+l} \tilde{\eta}_{i,j}(C_{\mathrm{usp}(n)})$ , which is considered as an element of  $\mathrm{End}((\mathbb{C}^{2n})^{\otimes k})$ . Recall that the diagonalization of these matrices will yield the joint moments of the entries of  $\tilde{g}$ , the matrix obtained from  $g$  by the map (1.2). Again, as a warm-up, let us compute the joint moments of order 2. If  $k = 2$ , the the eigenvectors and eigenvalues of  $M_{n,2}$  are:



eigenvalue	multiplicity	eigenvectors
$\frac{2n+1}{2}$	1	$\sum_{i=1}^n e[2i-1, 2i] - e[2i, 2i-1]$
$\frac{1}{2}$	$(n-1)(2n+1)$	$(e[2i-1, 2i] - e[2i, 2i-1]) - (e[2i+1, 2i+2] - e[2i+2, 2i+1]), i \leq n-1$ $e[2i-1, 2j-1] - e[2j-1, 2i-1], e[2i, 2j] - e[2j, 2i], 1 \leq i < j \leq n$ $e[2i-1, 2j] - e[2j, 2i-1], 1 \leq i \neq j \leq n$
$-\frac{1}{2}$	$n(2n+1)$	$e_k \otimes e_l + e_l \otimes e_k, 1 \leq k \leq l \leq 2n$

It follows that

$$\begin{aligned} \mathbb{E}[|g_{ii}|^2] &= \frac{1}{n} + \frac{n-1}{n} e^{-t} \quad ; \quad \mathbb{E}[|g_{ij}|^2] = \frac{1}{n} (1 - e^{-t}) \quad \forall i, j \in \llbracket 1, n \rrbracket ; \\ \mathbb{E}[(\tilde{g}_{ii})^2] &= e^{-\frac{n+1}{n}t} \quad ; \quad \mathbb{E}[(\tilde{g}_{ij})^2] = 0 \quad \forall i, j \in \llbracket 1, 2n \rrbracket . \end{aligned}$$

Now, for  $k=4$ , it is a little more tedious than before to find a complete list of “simple” eigenvectors of  $M_{n,4}$  (or at least a sufficient list to expand simple tensors). The list of possible eigenvalues of  $M_{n,4}$  is

$$\{2n+1, n+1, n, 3, 1, 0, -1, -3\},$$

and on the other hand, one can easily identify a basis of  $V_{2n+1}$ : it consists in the three vectors

$$\begin{aligned} v_{2n+1,1} &= \sum_{i,j=1}^n \begin{pmatrix} e[2i-1, 2i, 2j-1, 2j] + e[2i, 2i-1, 2j, 2j-1] \\ -e[2i, 2i-1, 2j-1, 2j] - e[2i-1, 2i, 2j, 2j-1] \end{pmatrix} ; \\ v_{2n+1,2} &= \sum_{i,j=1}^n \begin{pmatrix} e[2i-1, 2j-1, 2i, 2j] + e[2i, 2j, 2i-1, 2j-1] \\ -e[2i, 2j-1, 2i-1, 2j] - e[2i-1, 2j, 2i, 2j-1] \end{pmatrix} ; \\ v_{2n+1,3} &= \sum_{i,j=1}^n \begin{pmatrix} e[2i-1, 2j-1, 2j, 2i] + e[2i, 2j, 2j-1, 2i-1] \\ -e[2i, 2j-1, 2j, 2i-1] - e[2i-1, 2j, 2j-1, 2i] \end{pmatrix} . \end{aligned}$$

But then, it becomes really difficult to describe the other eigenspaces. However, one can still find<sup>17</sup> the eigenvector expansion of simple tensors such as  $e_i^{\otimes 4}$ ,  $e_i^{\otimes 2} e_j^{\otimes 2}$ , or  $e[i, j, k, l]$ ; hence, in the following, we just give these expansions (again it is easy to check that each part of an expansion is indeed an eigenvector). The tensor  $e[i, i, i, i]$  is an eigenvector in  $V_{-3}$ , so one obtains readily

$$\mathbb{E}[(\tilde{g}_{ii})^4] = e^{-\frac{2n+4}{n}t} \quad ; \quad \mathbb{E}[(\tilde{g}_{ij})^4] = \mathbb{E}[(\tilde{g}_{ii} \tilde{g}_{ij})^2] = \mathbb{E}[(\tilde{g}_{ij} \tilde{g}_{ik})^2] = 0.$$

On the other hand,  $e[2i-1, 2i-1, 2i, 2i]$  decomposes into the following eigenvectors

$$\frac{1}{2n(2n+1)} (v_{2n+1,2} + v_{2n+1,3})$$

-----

<sup>17</sup>To guess what these expansions were, we simply computed  $(M_{n,4})^r v$  for a sufficiently big number of distinct powers  $r$  of the matrix.

$$\begin{aligned}
& + \frac{n-2}{4n(n+1)} \sum_{\sigma \in S} \sum_{j \neq i} \left( \frac{e[2i-1, 2j-1, 2i, 2j] + e[2i, 2j, 2i-1, 2j-1]}{-e[2i-1, 2j, 2i, 2j-1] - e[2i, 2j-1, 2i-1, 2j]} \right)^\sigma \\
& + \frac{1}{4n(n+1)} \sum_{\sigma \in S} \sum_{j, k \neq i} \left( \frac{e[2j-1, 2k, 2j, 2k-1] + e[2j, 2k-1, 2j-1, 2k]}{-e[2j-1, 2k-1, 2j, 2k] - e[2j, 2k, 2j-1, 2k-1]} \right)^\sigma \\
& + \frac{n-1}{2n(n+1)} \left( \frac{2e[2i-1, 2i-1, 2i, 2i] + 2e[2i, 2i, 2i-1, 2i-1] - e[2i-1, 2i, 2i-1, 2i]}{-e[2i, 2i-1, 2i, 2i-1] - e[2i, 2i-1, 2i-1, 2i] - e[2i-1, 2i, 2i, 2i-1]} \right) \\
& \dots \\
& + \frac{1}{4(n+1)} \sum_{\sigma \in S} \sum_{j=1}^n \left( \frac{e[2i-1, 2j-1, 2i, 2j] + e[2i, 2j-1, 2i-1, 2j]}{-e[2i-1, 2j, 2i, 2j-1] - e[2i, 2j, 2i-1, 2j-1]} \right)^\sigma \\
& \dots \\
& + \frac{2n-1}{2(2n+1)(2n+2)} \sum_{\sigma \in S} \sum_{j \neq i} \left( \frac{e[2i-1, 2j, 2j-1, 2i] + e[2i, 2j-1, 2j, 2i-1]}{-e[2i-1, 2j-1, 2j, 2i] - e[2i, 2j, 2j-1, 2i-1]} \right)^\sigma \\
& + \frac{1}{2(2n+1)(2n+2)} \sum_{\sigma \in S} \sum_{j, k \neq i} \left( \frac{e[2j-1, 2k-1, 2j, 2k] + e[2j, 2k, 2j-1, 2k-1]}{-e[2j-1, 2k, 2j, 2k-1] - e[2j, 2k, 2j-1, 2k]} \right)^\sigma \\
& + \frac{(2n-1)(2n-2)}{6(2n+1)(2n+2)} \left( \frac{2e[2i-1, 2i-1, 2i, 2i] + 2e[2i, 2i, 2i-1, 2i-1] - e[2i-1, 2i, 2i-1, 2i]}{-e[2i, 2i-1, 2i, 2i-1] - e[2i, 2i-1, 2i-1, 2i] - e[2i-1, 2i, 2i, 2i-1]} \right) \\
& \dots \\
& + \frac{n-1}{2(n+1)} (e[2i-1, 2i-1, 2i, 2i] - e[2i, 2i, 2i-1, 2i-1]) \\
& + \frac{1}{4(n+1)} \sum_{\sigma \in S} \sum_{j \neq i} \left( \frac{e[2i-1, 2j, 2j-1, 2i] + e[2i, 2j, 2j-1, 2i-1]}{-e[2i-1, 2j-1, 2j, 2i] - e[2i, 2j-1, 2j, 2i-1]} \right)^\sigma \\
& \dots \\
& + \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_4} e[2i-1, 2i-1, 2i, 2i]^\sigma
\end{aligned}$$

with the parts of this expansion respectively in  $V_{2n+1}$ ,  $V_{n+1}$ ,  $V_n$ ,  $V_0$ ,  $V_{-1}$ , and  $V_{-3}$ . In these expansions,  $S = (\mathbb{Z}/2\mathbb{Z})^2$  denotes the group of permutations  $\{\text{id}, (1, 2), (3, 4), (1, 2)(3, 4)\}$ . As a consequence,

$$\begin{aligned}
\mathbb{E}[(\tilde{g}_{2i-1, 2i-1} \tilde{g}_{2i, 2i})^2] &= \frac{1}{n(2n+1)} + \frac{n-1}{n(n+1)} e^{-t} + \frac{1}{n+1} e^{-\frac{n+1}{n}t} + \frac{(2n-1)(2n-2)}{3(2n+1)(2n+2)} e^{-\frac{2n+1}{n}t} \\
&\quad + \frac{n-1}{2(n+1)} e^{-\frac{2n+2}{n}t} + \frac{1}{6} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[(\tilde{g}_{2i-1, 2i} \tilde{g}_{2i, 2i-1})^2] &= \frac{1}{n(2n+1)} + \frac{n-1}{n(n+1)} e^{-t} - \frac{1}{n+1} e^{-\frac{n+1}{n}t} + \frac{(2n-1)(2n-2)}{3(2n+1)(2n+2)} e^{-\frac{2n+1}{n}t} \\
&\quad - \frac{n-1}{2(n+1)} e^{-\frac{2n+2}{n}t} + \frac{1}{6} e^{-\frac{2n+4}{n}t}; \\
\mathbb{E}[(\tilde{g}_{2i-1, 2j-1} \tilde{g}_{2i, 2j})^2] &= \mathbb{E}[(\tilde{g}_{2i-1, 2j} \tilde{g}_{2i, 2j-1})^2] = \frac{1}{n(2n+1)} - \frac{1}{n(n+1)} e^{-t} + \frac{1}{(2n+1)(n+1)} e^{-\frac{2n+1}{n}t};
\end{aligned}$$

and the other moments of type  $\mathbb{E}[(\tilde{g}_{2i-1, a} \tilde{g}_{2i, b})^2]$  vanish. The other moments of type  $\mathbb{E}[(\tilde{g}_{ab} \tilde{g}_{cd})^2]$  with  $\{a, b\} \neq \{2i-1, 2i\}$  are much easier to compute: indeed, the expansion in eigenvectors of  $e[2i, 2i, 2j, 2j]$  is

$$\frac{1}{6} \left( \frac{2e[2i, 2i, 2j, 2j] + 2e[2j, 2j, 2i, 2i] - e[2i, 2j, 2i, 2j]}{-e[2j, 2i, 2j, 2i] - e[2i, 2j, 2j, 2i] - e[2j, 2i, 2i, 2j]} \right) + \frac{1}{2} \left( \frac{e[2i, 2i, 2j, 2j]}{-e[2j, 2j, 2i, 2i]} \right) + \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_4} e[2i, 2i, 2j, 2j]^\sigma$$

with each part respectively in  $V_0$ ,  $V_{-1}$  and  $V_{-3}$ ; and similarly for the expansions of  $e[2i-1, 2i-1, 2j, 2j]$  or  $e[2i-1, 2i-1, 2j-1, 2j-1]$ . Thus, assuming that  $\{a, b\}$  is not a pair  $\{2i-1, 2i\}$  in  $\llbracket 1, 2n \rrbracket$ , one has

$$\begin{aligned}\mathbb{E}[(\tilde{g}_{aa} \tilde{g}_{bb})^2] &= \frac{1}{3} e^{-\frac{2n+1}{n}t} + \frac{1}{2} e^{-\frac{2n+2}{n}t} + \frac{1}{6} e^{-\frac{2n+4}{n}t}; \\ \mathbb{E}[(\tilde{g}_{ab} \tilde{g}_{ba})^2] &= \frac{1}{3} e^{-\frac{2n+1}{n}t} - \frac{1}{2} e^{-\frac{2n+2}{n}t} + \frac{1}{6} e^{-\frac{2n+4}{n}t};\end{aligned}$$

and the other moments of type  $\mathbb{E}[(\tilde{g}_{ab} \tilde{g}_{cd})^2]$  with  $\{c, d\} \neq \{a, b\}$  vanish. The same expansions allows one to compute many moments of type  $\mathbb{E}[|g_{ij} g_{kl}|^2]$ , namely, all those that write as  $\mathbb{E}[|g_{ij} g_{ik}|^2]$ . For instance, since

$$|g_{ii}|^4 = (\tilde{g}_{2i-1, 2i-1} \tilde{g}_{2i, 2i} - \tilde{g}_{2i-1, 2i} \tilde{g}_{2i, 2i-1})^2,$$

its expectation is a linear combination of the expected values of  $(\tilde{g}_{2i-1, 2i-1} \tilde{g}_{2i, 2i})^2$ ,  $(\tilde{g}_{2i-1, 2i} \tilde{g}_{2i, 2i-1})^2$  and  $\tilde{g}_{2i-1, 2i-1} \tilde{g}_{2i-1, 2i} \tilde{g}_{2i, 2i} \tilde{g}_{2i, 2i-1}$ . This last expectation is

$$-\frac{1}{2n(2n+1)} - \frac{n-1}{2n(n+1)} e^{-t} - \frac{(2n-1)(2n-2)}{6(2n+1)(2n+2)} e^{-\frac{2n+1}{n}t} + \frac{1}{6} e^{-\frac{2n+4}{n}t}.$$

Thus, with a few more computations, one gets

$$\begin{aligned}\mathbb{E}[|g_{ii}|^4] &= \frac{3}{n(2n+1)} + \frac{3(n-1)}{n(n+1)} e^{-t} + \frac{(2n-1)(2n-2)}{(2n+1)(2n+2)} e^{-\frac{2n+1}{n}t}; \\ \mathbb{E}[|g_{ij}|^4] &= \frac{3}{n(2n+1)} - \frac{3}{n(n+1)} e^{-t} + \frac{3}{(2n+1)(n+1)} e^{-\frac{2n+1}{n}t}; \\ \mathbb{E}[|g_{ii} g_{ij}|^2] &= \frac{2}{n(2n+1)} + \frac{(n-2)}{n(n+1)} e^{-t} - \frac{2(2n-1)}{(2n+1)(2n+2)} e^{-\frac{2n+1}{n}t}; \\ \mathbb{E}[|g_{ij} g_{ik}|^2] &= \frac{2}{n(2n+1)} - \frac{2}{n(n+1)} e^{-t} + \frac{2}{(2n+1)(n+1)} e^{-\frac{2n+1}{n}t}.\end{aligned}$$

Finally, the expansion in eigenvectors of  $e[2i-1, 2i, 2j-1, 2j]$  enables one to compute the other moments of type  $\mathbb{E}[|g_{ij} g_{kl}|^2]$  which are useful in the case of quaternionic Grassmannians. As it is more than one page long, we skip the eigenvector expansion and just give the result:

$$\begin{aligned}\mathbb{E}[|g_{ii} g_{jj}|^2] &= \frac{2n-1}{n(n-1)(2n+1)} + \frac{2}{n+1} e^{-t} + \frac{n-3}{6(n-1)} e^{-\frac{2n-2}{n}t} \\ &\quad + \frac{n-2}{2n} e^{-2t} + \frac{2n^2-n+3}{3(n+1)(2n+1)} e^{-\frac{2n+1}{n}t}; \\ \mathbb{E}[|g_{ij} g_{ji}|^2] &= \frac{2n-1}{n(n-1)(2n+1)} - \frac{2}{n(n+1)} e^{-t} + \frac{n-3}{6(n-1)} e^{-\frac{2n-2}{n}t} \\ &\quad - \frac{n-2}{2n} e^{-2t} + \frac{2n^2-n+3}{3(n+1)(2n+1)} e^{-\frac{2n+1}{n}t}; \\ \mathbb{E}[|g_{ii} g_{jk}|^2] &= \frac{2n-1}{n(n-1)(2n+1)} + \frac{n^2-3n+1}{n(n+1)(n-2)} e^{-t} - \frac{n-3}{6(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\ &\quad - \frac{1}{2n} e^{-2t} - \frac{2n-3}{3(n+1)(2n+1)} e^{-\frac{2n+1}{n}t};\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[|g_{ij} g_{jk}|^2] &= \frac{2n-1}{n(n-1)(2n+1)} - \frac{2n-3}{n(n+1)(n-2)} e^{-t} - \frac{n-3}{6(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{1}{2n} e^{-2t} - \frac{2n-3}{3(n+1)(2n+1)} e^{-\frac{2n+1}{n}t}, \\
\mathbb{E}[|g_{ij} g_{kl}|^2] &= \frac{2n-1}{n(n-1)(2n+1)} - \frac{2n-2}{n(n+1)(n-2)} e^{-t} + \frac{1}{3(n-1)(n-2)} e^{-\frac{2n-2}{n}t} \\
&\quad + \frac{4}{3(n+1)(2n+1)} e^{-\frac{2n+1}{n}t}.
\end{aligned}$$

**Proposition 30.** *For the quaternionic Grassmannian varieties  $\text{Gr}(n, q, \mathbb{H})$  and the spaces  $\text{USp}(n)/\text{U}(n)$ , the coefficients of Lemma 26 are the following:*

$$\begin{aligned}
\text{Gr}(n, q, \mathbb{H}) : \quad & \frac{1}{2n^2 - n - 1} + \frac{n^2}{3} \left( \frac{1}{(n-1)(n-2)} - \frac{1}{pq(n-2)} \right) \phi^{(1^4, 0, \dots, 0)_n} \\
& + \frac{\frac{n^2}{pq} - 4}{(n-2)(n+1)} \phi^{(1^2, 0, \dots, 0)_{n-1}} + \frac{n^2}{3} \left( \frac{4}{(n+1)(2n+1)} + \frac{1}{pq(n+1)} \right) \phi^{(2, 2, 0, \dots, 0)_n}, \\
\text{USp}(n)/\text{U}(n) : \quad & \frac{1}{2n^2 + n} + \frac{4(n-1)(n+1)}{3n(2n+1)} \phi^{(2, 2, 0, \dots, 0)_n} + \frac{n+1}{3n} \phi^{(4, 0, \dots, 0)_n}.
\end{aligned}$$

*Proof.* The case of quaternionic Grassmannians is again done by using the expansion on top of page 64, with square modules instead of squares. One obtains the following formula for the expectation of  $(\phi^{(1, 1, 0, \dots, 0)_n})^2$ :

$$\begin{aligned}
& \frac{1}{2n^2 - n - 1} + \frac{\frac{n^2}{pq} - 4}{(n-2)(n+1)} e^{-t} + \frac{n^2}{3} \left( \frac{1}{(n-1)(n-2)} - \frac{1}{pq(n-2)} \right) e^{-\frac{2n-2}{n}t} \\
& + \frac{n^2}{3} \left( \frac{4}{(n+1)(2n+1)} + \frac{1}{pq(n+1)} \right) e^{-\frac{2n+1}{n}t},
\end{aligned}$$

hence the expansion in zonal functions by identification of the coefficients. Finally, for the spaces  $\text{USp}(n)/\text{U}(n)$ ,  $(\phi^{(2, 0, \dots, 0)_n})^2$  is equal to

$$\frac{1}{2n} (T[(\tilde{g}_{11})^4] + T[(\tilde{g}_{11}\tilde{g}_{22})^2] + T[(\tilde{g}_{12}\tilde{g}_{21})^2]) + \frac{n-1}{n} (T[(\tilde{g}_{13}\tilde{g}_{24})^2] + T[(\tilde{g}_{11}\tilde{g}_{33})^2] + T[(\tilde{g}_{13}\tilde{g}_{31})^2])$$

plus some remainder whose expectation under Brownian measures will be zero. Hence,

$$\mathbb{E}[(\phi^{(2, 0, \dots, 0)_n})^2] = \frac{1}{n(2n+1)} + \frac{4(n-1)(n+1)}{3n(2n+1)} e^{-\frac{2n+1}{n}t} + \frac{n+1}{3n} e^{-\frac{2n+4}{n}t},$$

and  $\frac{2n+1}{n}$  is the exponent corresponding to the spherical representation of label  $(2, 2, 0, \dots, 0)_n$ , whereas  $\frac{2n+4}{n}$  is the exponent corresponding to the spherical representation of label  $(4, 0, \dots, 0)_n$ .  $\square$

**4.2. Proof of the lower bound on the total variation distance.** The proof of the lower bound is now a simple application of Bienaymé-Chebyshev inequality. First, under the Haar measure, we have:

**Proposition 31.** *If  $E_a$  is the event  $\{|\Omega| \geq a\}$ , then the Haar measure of  $E_a$  satisfies the inequality*

$$\eta_X(E_a) \leq \frac{1}{a^2}$$

for every classical simple compact Lie group  $X = K$  and every classical simple compact symmetric space  $X = G/K$ .

*Proof.* The previous computations ensure that  $\mathbb{E}_\infty[|\Omega|^2] = 1$  in every case, so

$$\eta_X[|\Omega| \geq a] = \eta_X[|\Omega|^2 \geq a^2] \leq \frac{\mathbb{E}_\infty[|\Omega|^2]}{a^2} = \frac{1}{a^2}.$$

$\square$

Next, let us estimate  $\mathbb{E}_t[\Omega]$  and  $\text{Var}_t[\Omega]$  for  $t = \alpha(1 - \varepsilon) \log n$ . The exact values are listed in the following table:

$K$ or $G/K$	$\mathbb{E}_t[\Omega]$	$\text{Var}_t[\Omega]$
$\text{SO}(n)$	$n e^{-\frac{n-1}{2n}t}$	$1 + \frac{n(n-1)}{2} e^{-\frac{n-4}{n}t} + \left(\frac{n(n+1)}{2} - 1\right) e^{-t} - n^2 e^{-\frac{n-1}{n}t}$
$\text{SU}(n)$	$n e^{-\frac{n^2-1}{2n^2}t}$	$1 + (n^2 - 1) e^{-t} - n^2 e^{-\frac{n^2-1}{n^2}t}$
$\text{USp}(n)$	$2n e^{-\frac{2n+1}{4n}t}$	$1 + (2n+1)(n-1) e^{-t} + (2n+1)n e^{-\frac{n+1}{n}t} - 4n^2 e^{-\frac{2n+1}{2n}t}$
$\text{Gr}(n, q, \mathbb{R})$	$\sqrt{\frac{(n+2)(n-1)}{2}} e^{-t}$	$1 + \left(\frac{2n^2}{pq} - 8\right) \frac{(n-1)(n+2)}{(n-2)(n+4)} e^{-t} + \frac{n^2}{3} \left(\frac{n+2}{n-2} - \frac{(n+2)(n-1)}{pq(n-2)}\right) e^{-\frac{2n-2}{n}t}$ $+ \frac{n^2}{6} \left(\frac{n-1}{n+4} + \frac{2(n+2)(n-1)}{pq(n+4)}\right) e^{-\frac{2n+4}{n}t} - \frac{(n+2)(n-1)}{2} e^{-2t}$
$\text{Gr}(n, q, \mathbb{C})$	$\sqrt{n^2 - 1} e^{-t}$	$1 + \left(\frac{2n^2}{pq} - 8\right) \frac{n^2-1}{n^2-4} e^{-t} + \frac{n^2}{2} \left(\frac{n+1}{n-2} - \frac{n^2-1}{pq(n-2)}\right) e^{-\frac{2n-2}{n}t}$ $+ \frac{n^2}{2} \left(\frac{n-1}{n+2} + \frac{n^2-1}{pq(n+2)}\right) e^{-\frac{2n+2}{n}t} - (n^2 - 1) e^{-2t}$
$\text{Gr}(n, q, \mathbb{H})$	$\sqrt{(2n+1)(n-1)} e^{-t}$	$1 + \left(\frac{n^2}{pq} - 4\right) \frac{(n-1)(2n+1)}{(n-2)(n+1)} e^{-t} + \frac{n^2}{3} \left(\frac{2n+1}{n-2} - \frac{(2n+1)(n-1)}{pq(n-2)}\right) e^{-\frac{2n-2}{n}t}$ $+ \frac{n^2}{3} \left(\frac{4(n-1)}{(n+1)} + \frac{(2n+1)(n-1)}{pq(n+1)}\right) e^{-\frac{2n+1}{n}t} - (2n+1)(n-1) e^{-2t}$
$\text{SO}(2n)/\text{U}(n)$	$\sqrt{n(2n-1)} e^{-\frac{n-1}{n}t}$	$1 + \frac{(n-1)(2n-1)}{3} e^{-\frac{2n-4}{n}t} + \frac{4(n^2-1)}{3} e^{-\frac{2n-1}{n}t} - n(2n-1) e^{-\frac{2n-2}{n}t}$
$\text{SU}(n)/\text{SO}(n)$	$\sqrt{\frac{n(n+1)}{2}} e^{-\frac{(n-1)(n+2)}{n^2}t}$	$1 + \frac{(n+2)(n-1)}{2} e^{-\frac{2n+2}{n}t} - \frac{n(n+1)}{2} e^{-\frac{(n-1)(2n+4)}{n^2}t}$
$\text{SU}(2n)/\text{USp}(n)$	$\sqrt{2n^2 - n} e^{-\frac{(n-1)(2n+1)}{2n^2}t}$	$1 + (2n^2 - n - 1) e^{-\frac{2n-1}{n}t} - (2n^2 - n) e^{-\frac{(n-1)(2n+1)}{n^2}t}$
$\text{USp}(n)/\text{U}(n)$	$\sqrt{n(2n+1)} e^{-\frac{n+1}{n}t}$	$1 + \frac{4(n-1)(n+1)}{3} e^{-\frac{2n+1}{n}t} + \frac{(2n+1)(n+1)}{3} e^{-\frac{2n+4}{n}t} - n(2n+1) e^{-\frac{2n+2}{n}t}$

In the following we assume  $\varepsilon < \frac{1}{4}$ ; indeed, Lemma 5 ensures that it is sufficient to control the total variation distance around the cut-off time. We shall use a lot the inequality of convexity

$$\exp(x) \leq 1 + \frac{e^y - 1}{y} x \quad \forall x \in (0, y).$$

**Lemma 32.** *Under the usual assumptions on  $n$ , for groups and spaces of structures (but not for Grassmannian varieties),  $\text{Var}_t[\Omega]$  is uniformly bounded for every  $t = \alpha(1 - \varepsilon) \log n$  with  $\varepsilon \in (0, 1/4)$ . Possible upper bounds are listed below:*

$$\begin{aligned} \text{SU}(n), \text{SU}(n)/\text{SO}(n), \text{SU}(2n)/\text{USp}(n) &: 1 \quad ; \\ \text{SO}(2n)/\text{U}(n), \text{USp}(n), \text{USp}(n)/\text{U}(n) &: 3 \quad ; \\ \text{SO}(n) &: 8. \end{aligned}$$

*Proof.* We proceed case by case, and denote  $\Delta_t(\lambda, \mu) = e^{-\lambda t} - e^{-\mu t}$ . Notice that  $\Delta_t(\lambda, \mu) \leq 0$  if  $\lambda \geq \mu$ . On the other hand,  $\Delta_t(\lambda, \mu)$  is always smaller than 1 for  $\lambda, \mu \geq 0$ .

- $\text{SO}(n)$ :

$$\begin{aligned}\text{Var}_t[\Omega] &= \Delta_t\left(0, \frac{n-1}{n}\right) + \frac{n(n-1)}{2} \Delta_t\left(\frac{n-4}{n}, \frac{n-1}{n}\right) + \left(\frac{n(n+1)}{2} - 1\right) \Delta_t\left(1, \frac{n-1}{n}\right) \\ &\leq 1 + \frac{n(n-1)}{2} \Delta_t\left(\frac{n-4}{n}, \frac{n-1}{n}\right) = 1 + \frac{n(n-1)}{2} e^{-\frac{n-1}{n}t} \left(e^{\frac{3t}{n}} - 1\right) \\ &\leq 1 + \frac{13}{2} n \log n e^{-\frac{n-1}{n}t}\end{aligned}$$

since  $\frac{6 \log n}{n} \leq 1.382$  when  $n \geq 10$ , and  $\frac{e^{1.382}-1}{1.382} \leq \frac{13}{6}$ . Then,

$$e^{-\frac{n-1}{n}t} \leq e^{-\frac{3(n-1) \log n}{2n}} = n^{-1} e^{-\frac{(n-3) \log n}{2n}} \leq \frac{14}{13} (n \log n)^{-1}$$

for  $n \geq 10$ , so  $\text{Var}_t[\Omega] \leq 1 + 7 = 8$ .

- $\text{SU}(n)$ :

$$\text{Var}_t[\Omega] = \Delta_t\left(0, \frac{n^2-1}{n^2}\right) + (n^2-1) \Delta_t\left(1, \frac{n^2-1}{n^2}\right) \leq 1.$$

- $\text{USp}(n)$ :

$$\begin{aligned}\text{Var}_t[\Omega] &= \Delta_t\left(0, \frac{2n+1}{2n}\right) + (2n+1)(n-1) \Delta_t\left(1, \frac{2n+1}{2n}\right) + (2n+1)n \Delta_t\left(\frac{2n+2}{2n}, \frac{2n+1}{2n}\right) \\ &\leq 1 + (2n+1)(n-1) \Delta_t\left(1, \frac{2n+1}{2n}\right) \leq 1 + 2n^2 e^{-\frac{2n+1}{2n}t} \left(e^{\frac{t}{2n}} - 1\right) \\ &\leq 1 + \frac{5}{2} n \log n e^{-\frac{2n+1}{2n}t}\end{aligned}$$

since  $\frac{\log n}{n} \leq 0.367$  when  $n \geq 3$ , and  $\frac{e^{0.367}-1}{0.367} \leq \frac{5}{4}$ . Then,

$$e^{-\frac{2n+1}{2n}t} \leq e^{-\frac{3 \log n}{2}} = n^{-\frac{3}{2}} \leq \frac{4}{5} (n \log n)^{-1}$$

for  $n \geq 3$ , so  $\text{Var}_t[\Omega] \leq 1 + 2 = 3$ .

- $\text{SO}(2n)/\text{U}(n)$ :

$$\begin{aligned}\text{Var}_t[\Omega] &= \Delta_t\left(0, \frac{2n-2}{n}\right) + \frac{(n-1)(2n-1)}{3} \Delta_t\left(\frac{2n-4}{n}, \frac{2n-2}{n}\right) + \frac{4(n^2-1)}{3} \Delta_t\left(\frac{2n-1}{n}, \frac{2n-2}{n}\right) \\ &\leq 1 + \frac{(n-1)(2n-1)}{3} \Delta_t\left(\frac{2n-4}{n}, \frac{2n-2}{n}\right) \leq 1 + \frac{2n^2}{3} e^{-\frac{2n-2}{n}t} \left(e^{\frac{2t}{n}} - 1\right) \\ &\leq 1 + \frac{20}{9} n \log 2n e^{-\frac{2n-2}{n}t}\end{aligned}$$

since  $\frac{2 \log 2n}{n} \leq 0.922$  when  $2n \geq 10$ , and  $\frac{e^{0.922}-1}{0.922} \leq \frac{5}{3}$ . Since

$$e^{-\frac{2n-2}{n}t} \leq e^{-\frac{3(n-1) \log 2n}{2n}} = \frac{1}{2} n^{-1} e^{-\frac{(n-3) \log 2n}{2n}} \leq \frac{3}{4} (n \log 2n)^{-1}$$

for  $2n \geq 10$ , one concludes that  $\text{Var}_t[\Omega] \leq 1 + \frac{5}{3} \leq 3$ .

- $\text{SU}(n)/\text{SO}(n)$ :

$$\text{Var}_t[\Omega] = \Delta_t\left(0, \frac{2(n-1)(n+2)}{n^2}\right) + \frac{(n+2)(n-1)}{2} \Delta_t\left(\frac{2(n+1)}{n}, \frac{2(n-1)(n+2)}{n^2}\right) \leq 1.$$

- $SU(2n)/USp(n)$ :

$$\text{Var}_t[\Omega] = \Delta_t\left(0, \frac{(n-1)(2n+1)}{n^2}\right) + (2n^2 - n - 1) \Delta_t\left(\frac{2n-1}{n}, \frac{(n-1)(2n+1)}{n^2}\right) \leq 1.$$

- $USp(n)/U(n)$ :

$$\begin{aligned} \text{Var}_t[\Omega] &= \Delta_t\left(0, \frac{2n+2}{n}\right) + \frac{4(n^2-1)}{3} \Delta_t\left(\frac{2n+1}{n}, \frac{2n+2}{n}\right) + \frac{2n^2+3n+1}{3} \Delta_t\left(\frac{2n+4}{n}, \frac{2n+2}{n}\right) \\ &\leq 1 + \frac{4(n^2-1)}{3} \Delta_t\left(\frac{2n+1}{n}, \frac{2n+2}{n}\right) \leq 1 + \frac{4n^2}{3} e^{-\frac{2n+2}{n}t} \left(e^{\frac{t}{n}} - 1\right) \\ &\leq 1 + \frac{5}{3} n \log n e^{-\frac{2n+2}{n}t} \end{aligned}$$

by using the same estimate on  $\frac{\log n}{n}$  as in the case of  $USp(n)$ . Since

$$e^{-\frac{2n+2}{n}t} \leq e^{-\frac{3 \log n}{2}} = n^{-\frac{3}{2}} \leq \frac{4}{5} (n \log n)^{-1}$$

for  $n \geq 3$ , one obtains  $\text{Var}_t[\Omega] \leq 1 + \frac{4}{3} \leq 3$ .

It is not possible to prove such uniform bounds for Grassmannians, because of the term  $e^{-t}$  that appears in the variance. We shall address this problem in Lemma 34.  $\square$

**Proposition 33.** Denote  $K_X$  the bound computed in the previous Lemma for the variance of the discriminating zonal function  $\Omega$  associated to a space  $X$ . Then,

$$d_{TV}(\mu_t, \text{Haar}) \geq 1 - \frac{4(K_X + 1)}{(\mathbb{E}_t[\Omega])^2}.$$

*Proof.* Assuming  $a$  smaller than  $m = \mathbb{E}_t[\Omega]$ , if  $|\Omega - m| \leq a$ , then  $|\Omega| \geq m - a$ . Consequently,

$$\mu_t[|\Omega| \geq m - a] \geq 1 - \mathbb{P}[|\Omega - m| > a] \geq 1 - \frac{\text{Var}_t[\Omega]}{a^2} = 1 - \frac{K_X}{a^2}.$$

Next, take  $a = \frac{m}{2}$ . The combination of Lemma 31 and of the previous inequality yields

$$d_{TV}(\mu_t, \text{Haar}) \geq \mu_t(E_a) - \eta_X(E_a) \geq 1 - \frac{K_X + 1}{a^2} = 1 - \frac{4(K_X + 1)}{m^2}.$$

Since  $m^2$  behaves as  $n^{2\varepsilon}$ , this essentially ends the proof of the lower bounds in the case of compact Lie groups and compact spaces of structures. More precisely:

- $SO(n)$ :  $m^2 \geq n^{2\varepsilon}$  so the constant  $c$  in our main Theorem 8 is  $4(8+1) = 36$ .
- $SU(n)$ : again,  $m^2 \geq n^{2\varepsilon}$ , so the constant is  $4(1+1) = 8$ .
- $USp(n)$ : here,  $m^2 \geq 4n^{2\varepsilon} e^{-\frac{\log n}{2n}} \geq \frac{16}{5} n^{2\varepsilon}$  for  $n \geq 3$ , so the constant is  $\frac{5}{16} 4(3+1) = 5$ .
- $SO(2n)/U(n)$ :  $m^2 \geq \frac{2n-1}{4n} (2n)^{2\varepsilon} \geq \frac{9}{20} (2n)^{2\varepsilon}$  for  $2n \geq 10$ , whence a constant  $\frac{9}{20} 4(3+1) = \frac{36}{5} \leq 8$ .
- $SU(n)/SO(n)$ :  $m^2 \geq \frac{n^{2\varepsilon}}{2} e^{-\frac{2(n-2)\log n}{n^2}} \geq \frac{n^{2\varepsilon}}{3}$  for  $n \geq 2$ , so a possible constant is  $3 \times 4(1+1) = 24$ .
- $SU(2n)/USp(n)$ :  $m^2 \geq \frac{2n-1}{4n} (2n)^{2\varepsilon} \geq \frac{3}{8} (2n)^{2\varepsilon}$ , and a possible constant is  $\frac{8}{3} 4(1+1) = \frac{64}{3} \leq 22$ .
- $USp(n)/U(n)$ :  $m^2 \geq 2n^{2\varepsilon} e^{-\frac{2 \log n}{n}} \geq \frac{16}{17} n^{2\varepsilon}$  for  $n \geq 3$ , whence a constant  $\frac{17}{16} 4(3+1) = 17$ .

$\square$

Unfortunately, for Grassmannian varieties, the variance of  $\Omega$  at time  $t = (1 - \varepsilon) \log n$  can only be bounded by a constant times  $n^\varepsilon$ . However, since the mean of  $\Omega$  is also of order  $n^\varepsilon$ , this will still ensure that the discriminating zonal spherical function has not at all the same behavior under Haar measure

and under Brownian measures before cut-off time. The only downside is the loss of a factor  $n^\varepsilon$  in the estimate of the total variation distance.

**Lemma 34.** *Under the usual assumptions on  $n$ , for Grassmannian varieties,*

$$\frac{\text{Var}_t[\Omega]}{n^\varepsilon} \leq \begin{cases} 3 & \text{if } \mathbb{k} = \mathbb{R}, \\ 5 & \text{if } \mathbb{k} = \mathbb{C} \text{ or } \mathbb{H}, \end{cases}$$

for every  $t = \alpha(1 - \varepsilon) \log n$  with  $\varepsilon \in (0, 1/4)$ .

*Proof.* The quantity  $\frac{1}{pq}$  is bounded by

$$\frac{4}{n^2} \leq \frac{1}{pq} \leq \frac{1}{n-1},$$

the extremal values corresponding to  $p = q = \frac{n}{2}$  and to  $p = n-1$  or  $q = n-1$ . In particular, in the expansions hereafter, all the coefficients that precede differences of exponentials  $\Delta_t(\lambda, \mu)$  are positive. Now, we proceed case by case:

- $\text{Gr}(n, q, \mathbb{R})$ :

$$\begin{aligned} \text{Var}_t[\Omega] &= \Delta_t(0, 2) + \left( \frac{2n^2}{pq} - 8 \right) \frac{(n-1)(n+2)}{(n-2)(n+4)} \Delta_t(1, 2) + \frac{n^2}{3} \left( \frac{n+2}{n-2} - \frac{(n+2)(n-1)}{pq(n-2)} \right) \Delta_t\left(\frac{2n-2}{n}, 2\right) \\ &\quad + \frac{n^2}{6} \left( \frac{n-1}{n+4} + \frac{2(n+2)(n-1)}{pq(n+4)} \right) \Delta_t\left(\frac{2n+4}{n}, 2\right) \\ &\leq 1 + 2n \Delta_t(1, 2) + \frac{n^2}{3} \Delta_t\left(\frac{2n-2}{n}, 2\right). \end{aligned}$$

For the difference  $\Delta_t(1, 2)$ , one cannot obtain a better bound than  $e^{-t} = n^{\varepsilon-1}$ . The second difference  $\Delta_t\left(\frac{2n-2}{n}, 2\right)$  is bounded from above by

$$e^{-2t} \left( e^{\frac{2t}{n}} - 1 \right) \leq n^{-\frac{3}{2}} \frac{8 \log n}{3n} \leq 2n^{-2}$$

by using similar arguments as in the proof of Lemma 32, and the inequality  $n \geq 10$ . So,

$$\text{Var}_t[\Omega] \leq 1 + \frac{2}{3} + 2n^\varepsilon \leq 3n^\varepsilon.$$

- $\text{Gr}(n, q, \mathbb{C})$ :

$$\begin{aligned} \text{Var}_t[\Omega] &= \Delta_t(0, 2) + \left( \frac{2n^2}{pq} - 8 \right) \frac{n^2-1}{n^2-4} \Delta_t(1, 2) + \frac{n^2}{2} \left( \frac{n+1}{n-2} - \frac{n^2-1}{pq(n-2)} \right) \Delta_t\left(\frac{2n-2}{n}, 2\right) \\ &\quad + \frac{n^2}{2} \left( \frac{n-1}{n+2} + \frac{n^2-1}{pq(n+2)} \right) \Delta_t\left(\frac{2n+2}{n}, 2\right) \\ &\leq 1 + 2n \Delta_t(1, 2) + \frac{n^2}{2} \Delta_t\left(\frac{2n-2}{n}, 2\right). \end{aligned}$$

The second difference is controlled exactly as in the case of real Grassmannians, but under the constraint  $n \geq 2$ :

$$\Delta_t\left(\frac{2n-2}{n}, 2\right) \leq e^{-2t} \left( e^{\frac{2t}{n}} - 1 \right) \leq n^{-\frac{3}{2}} \frac{3 \log n}{n} \leq \frac{9}{4} n^{-2}.$$

Hence,  $\text{Var}_t[\Omega] \leq 1 + \frac{9}{8} + 2n^\varepsilon \leq 5n^\varepsilon$ .



- $\text{Gr}(n, q, \mathbb{H})$ :

$$\begin{aligned} \text{Var}_t[\Omega] &= \Delta_t(0, 2) + \frac{n^2}{3} \left( \frac{2n+1}{n-2} - \frac{(2n+1)(n-1)}{pq(n-2)} \right) \Delta_t\left(\frac{2n-2}{n}, 2\right) \\ &\quad + \left( \frac{n^2}{pq} - 4 \right) \frac{(n-1)(2n+1)}{(n-2)(n+1)} \Delta_t(1, 2) + \frac{n^2}{3} \left( \frac{4(n-1)}{(n+1)} + \frac{(2n+1)(n-1)}{pq(n+1)} \right) \Delta_t\left(\frac{2n+1}{n}, 2\right) \\ &\leq 1 + 2n \Delta_t(1, 2) + \frac{2n^2}{3} \Delta_t\left(\frac{2n-2}{n}, 2\right) \leq 1 + 2n^\varepsilon + \frac{3}{2} \leq 5n^\varepsilon. \end{aligned}$$

□

Now, Proposition 33 still holds, but with  $K_X$  varying with  $n$  and equal to  $3n^\varepsilon$  or  $5n^\varepsilon$  according to the field  $\mathbb{k} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Thus:

**Proposition 35.** *For Grassmannian varieties  $\text{Gr}(n, q, \mathbb{k})$ , if  $t = (1 - \varepsilon) \log n$  with  $\varepsilon \in (0, 1/4)$ , then*

$$d_{\text{TV}}(\mu_t, \text{Haar}) \geq 1 - \frac{Ln^\varepsilon}{m^2} \quad \text{with } L = \begin{cases} 16 & \text{if } \mathbb{k} = \mathbb{R}, \\ 24 & \text{if } \mathbb{k} = \mathbb{C} \text{ or } \mathbb{H}. \end{cases}$$

Finally, the deduction of the constants in Theorem 8 goes as follows:

- $\text{Gr}(n, q, \mathbb{R})$ :  $m \geq \frac{n^{2\varepsilon}}{2}$ , so the constant can be taken equal to  $2 \times 16 = 32$ .
- $\text{Gr}(n, q, \mathbb{C})$ :  $m \geq \frac{n^2-1}{n^2} n^{2\varepsilon} \geq \frac{3}{4} n^{2\varepsilon}$ , so a possible constant is again  $\frac{4}{3} 24 = 32$ .
- $\text{Gr}(n, q, \mathbb{H})$ :  $m \geq \frac{2n^2-n-1}{2} n^{2\varepsilon} \geq \frac{3}{2} n^{2\varepsilon}$  for  $n \geq 3$ , whence a constant  $\frac{2}{3} 24 = 16$ .

These computations end the proof of the cut-off phenomenon.

## REFERENCES

- [AD86] D. Aldous and P. Diaconis. Shuffling cards and stopping times. *Amer. Math. Monthly*, 93(5):333–348, 1986.
- [App11] D. Applebaum. Infinitely divisible central probability measures on compact Lie groups — Regularity, semigroups and transition kernels. *Ann. Probab.*, 39(6):2474–2496, 2011.
- [BD85] T. Bröcker and T. Dieck. *Representations of Compact Lie Groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, 1985.
- [BD92] D. Bayer and P. Diaconis. Trailing the dovetail shuffle to its lair. *Ann. Appl. Probab.*, 2(2):294–313, 1992.
- [CSC08] G.-Y. Chen and L. Saloff-Coste. The cutoff phenomenon for ergodic Markov processes. *Electronic J. Probab.*, 13(3):26–78, 2008.
- [CSST08] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli. *Harmonic Analysis on Finite Groups*, volume 108 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2008.
- [Dia96] P. Diaconis. The cutoff phenomenon in finite Markov chains. *Proc. Nat. Acad. Sci. of U.S.A.*, 93(4):1659–1664, 1996.
- [DS94] P. Diaconis and M. Shahshahani. On the eigenvalues of random matrices. *Journal of Applied Probability*, 31:49–62, 1994.
- [DSC96] P. Diaconis and L. Saloff-Coste. Random walks on finite groups: A survey of analytic techniques. In H. Heyer, editor, *Prob. Meas. on Groups XI*, pages 44–75. World Scientific Singapore, 1996.
- [Far08] J. Faraut. *Analysis on Lie Groups: An Introduction*, volume 110 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2008.
- [FH91] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, 1991.
- [Gas70] G. Gasper. Linearization of the product of Jacobi polynomials. I and II. *Can. J. Math.*, 22:171–175; 582–593, 1970.
- [GW09] R. Goodman and N. R. Wallach. *Symmetry, Representations, and Invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer-Verlag, 2009.
- [Hel78] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, 1978.
- [Hel84] S. Helgason. *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*. Academic Press, 1984.

- [HS94] G. Heckman and H. Schlichtkrull. *Harmonic Analysis and Special Functions on Symmetric Spaces*, volume 16 of *Perspectives in Mathematics*. Academic Press, 1994.
- [Hun56] G. A. Hunt. Semigroups of measures on Lie groups. *Trans. Am. Math.*, 81:264–293, 1956.
- [Lév11] T. Lévy. Asymptotics of Brownian motions on classical Lie groups, the master field on the plane, and the Makeenko-Migdal equations. [arXiv:1112.2452v1 \[math-ph\]](#), 2011.
- [Lia04a] M. Liao. Lévy processes and Fourier analysis on compact Lie groups. *Ann. Probab.*, 32(2):1553–1573, 2004.
- [Lia04b] M. Liao. *Lévy processes in Lie groups*, volume 162 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 2004.
- [MOT53] W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions*. McGraw-Hill Book Company, 1953.
- [Por96a] U. Porod. The cut-off phenomenon for random reflections. *Ann. Probab.*, 24(1):74–96, 1996.
- [Por96b] U. Porod. The cut-off phenomenon for random reflections II: complex and quaternionic cases. *Probab. Th. Rel. Fields*, 104(2):181–209, 1996.
- [Ros94] J. S. Rosenthal. Random rotations: characters and random walks on  $SO(N)$ . *Ann. Probab.*, 22:398–423, 1994.
- [SC94] L. Saloff-Coste. Precise estimates on the rate at which certain diffusions tend to equilibrium. *Math. Zeitschrift*, 217:641–677, 1994.
- [SC04] L. Saloff-Coste. On the convergence to equilibrium of Brownian motion on compact simple Lie groups. *J. Geometric Analysis*, 14(4):715–733, 2004.
- [ST97] E. B. Saff and V. Totik. *Logarithmic Potentials with External Fields*, volume 316 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1997.
- [Sze39] G. Szegő. *Orthogonal Polynomials*, volume 23 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, 1939.
- [Var89] V. S. Varadarajan. *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, volume 16 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1989.